James Thornton

ELEMENTS

OF

GEOMETRY,

WITH

PRACTICAL APPLICATIONS,

FOR THE USE OF SCHOOLS.

BY T[imothy]. WALKER,

TEACHER OF MATHEMATICS IN THE ROUND HILL SCHOOL,
AT NORTHAMPTON, MASS.

THIRD EDITION, IMPROVED.

BOSTON:
PUBLISHED BY RICHARDSON, LORD & HOLBROOK,
133 WASHINGTON STREET.
1831.

WAITT & DOW'S PRESS,
122 Washington-Street.

PREFACE TO THE FIRST EDITION.

In preparing the following work, two objects have been kept constantly in view. First, I have endeavored to bring the essential principles of Geometry within a small compass; and, secondly, to make their connexion easy to be understood. That such a book is wanted, I am convinced from personal experience. The works of Euclid and Legendre, the two most generally studied in New-England, though each is nearly perfect in its kind, are, for that very reason, suited only to the highest seminaries of learning. They cost too much and they require too much time, to be generally studied in academies and schools. Moreover, they are too abstruse and difficult for the comprehension of very young pupils. All this is a necessary consequence of their fullness and perfection, as treatises on this branch of Mathematics. They necessarily contain many propositions, which are not requisite for the understanding of subsequent branches, such as Trigonometry and Conic Sections; and which are not made use of in the more important practical applications, such as Mensuration, Surveying and Navigation. To study them would be an excellent discipline for the mind, if there were time: but this detains the pupil too long from the subsequent higher branches, which afford an equally salutary discipline for the mind, and, in addition to this, are absolutely essential to a complete practical education.

Under these impressions, I have omitted all such propositions as are not absolutely necessary for the understanding of the subsequent parts of a mathemat-

ical course. I have condensed those which I have admitted, as much as was compatible with clearness and perspicuity, that the book might be small and consequently cheap. I have placed the problems immediately after the theorems upon which they depend, that this dependence might always be readily perceived. I have avoided the general use of the technical terms, *problem*, *theorem*, *corollary*, *scholium*, and *axiom*, from conviction that they confuse rather than assist young minds; and have used instead of them, the general term *proposition*. With regard to *definitions*, I have, for the most part, deferred giving them, until the magnitudes or figures defined were to be immediately considered, believing that in this way they would be more readily understood and remembered. Whenever I have ventured to depart from the definitions in common use, as in the case of a *straight line* and of *parallel lines*, it has been done, not for the sake of being original, but solely with a view to greater simplicity; remembering that the work was for youth and not for adepts. The same remark applies to those demonstrations which are believed to be original, such as the *equality of the angles formed by parallel lines meeting a straight line;* and *the approximate ratio of the circumference of a circle to its diameter;* also, *several of the properties of a triangle by inscribing it in a circle.*

The division of the work into three sections, is founded in the nature of the subject. Extension, or the space which matter occupies, has three dimensions, length, breadth, and thickness. These may be considered separately or in connexion. When we consider *length alone*, its representative is a *line*. Hence *the first section treats of lines and their relations.* When we consider *length and breadth together*, or *length in two ways*, their representative is a *surface*. Hence *the second section treats of surfaces.* Lastly, when we consider *length, breadth, and thickness together*, or *length in three ways*, their representative is a *solid*. Hence *the third section treats of solids.* The *appendix* is not designed to give a complete view of the applications of geometry to practical purposes, for this would require a separate volume; but only to give the pupil a general notion of the uses of geometry, by

some of the most important particular cases. *Questions* are placed at the end of the whole, because it is believed they will assist young pupils in reviewing. Those propositions and definitions which are thought proper to be *committed to memory* by the pupil, are printed in *Italics*, and separated from the context by a *dash* at the beginning and end.

It is proper here to observe that the *circle* is uniformly treated in the following work, as a *regular polygon of an infinite number of sides*. This has done more than all other expedients, to reduce the dimensions of the work, without diminishing the number of results. If this principle had not been introduced, and the properties of the circle and figures depending upon it had been demonstrated by the usual method of a *reductio ad absurdum*, at least *thirty pages* more would have been necessary to obtain the same results as are here obtained. This appeared to be a sufficient reason for introducing it.

Under the impression that every student, who is at all inquisitive or curious, must desire to know something of the *history* of Geometry, its origin and progress are briefly traced in the *Introduction*. If the student should read this before studying the body of the work, it is recommended that he read it again, after he has finished the course of demonstration.

I shall make but one observation more. This work is prepared for *young pupils*, and does not *profess* to be a complete treatise on all the elements of Geometry. If, therefore, it be honored with criticism, it is but just that these things should be kept in mind. Its pretensions are humble; and that it has many faults, no one can be more sensible than

THE AUTHOR.

Round Hill, Northampton, Feb. 2, 1829.

PREFACE TO THE SECOND EDITION.

The speedy sale of the first edition of this work, has justified the belief of the author, that such a work was wanted. In placing the present edition before the public, it is only necessary to notice some slight changes, which have been made. The technical terms, *problem*, *theorem*, *corollary*, *scholium*, and *axiom*, which were not adopted in the first edition, are used in this. The form of reasoning is rendered rather more synthetic, by uniformly placing the propositions or definitions at the commencement of the sections. The proportions are placed in lines by themselves, that their connexion may more readily be perceived by the eye. Some demonstrations have been enlarged for the sake of greater clearness. In the figures on the plates, some of the lines are made larger than the rest, for the purpose of rendering the order of construction more obvious. On the whole, it is believed that these changes will be found to be improvements. They have been suggested, not only by the experience of the author, in teaching from the book, but also by several friends, whose intelligence gives weight to their opinion.

Cambridge, Nov. 4th, 1829.

INTRODUCTION,

CONTAINING

A BRIEF HISTORY OF GEOMETRY.

GEOMETRY takes its name from two Greek words signifying *the measure of land*, this being the first purpose to which it was applied. It is generally supposed to have originated in Egypt, and to have owed its invention to the necessity of determining anew every year, the land-marks which designated the shares of land belonging to each proprietor, when the annual inundations of the Nile had obliterated or removed them. This however is conjecture. But it is known with certainty, that the Egyptians had some little knowledge of the first principles of Geometry.

The scanty knowledge of the Egyptians was brought into Greece by Thales the Philosopher, about 640 years before Christ; and there, Geometry grew up, from a few scattered elements, into that exact and beautiful science which it now is. While in Egypt, it is said that Thales learned enough of Geometry to enable him to measure the heights of the pyramids by means of their shadows, and to ascertain the distance of vessels remote from the shore. Upon his return to Greece, he not only encouraged the study among his countrymen, but made some important discoveries himself. He first found out that *all the angles inscribed in a semicircle are right angles*, and was so delighted with the discovery that he made a sacrifice to the Muses.

Soon after Thales came Anaxagoras. He was imprisoned on account of his opinions respecting astronomy, and during his confinement employed himself in attempting to find the *quadrature of the circle, or the*

ratio of the circumference to the diameter. It is remarkable that the first attempt to solve the most famous problem in Geometry, should have been a prison amusement.

Pythagoras was born about 580 years before Christ. After having travelled into Egypt and India, he gave himself up to the study of Geometry with wonderful ardor and success. It was he who discovered that *the square of the hypothenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.* To express his joy and gratitude for this great discovery, we are told that he sacrificed one hundred oxen to the Muses. He also discovered that *the circle is the greatest of all figures of the same perimeter.*

The first man who digested the Elements of Geometry into a regular treatise, was Hippocrates, who lived soon after Pythagoras. This work has not come down to us; but history informs us, respecting Hippocrates, that he was originally a merchant; that he visited Athens on business, and was one day tempted by mere curiosity to visit the schools of philosophy; that he there heard of Geometry for the first time, and was so charmed that he renounced all other pursuits and gave his whole mind to this. No wonder that with such fervent devotion to the study, he soon became one of the best Geometers of his time.

We now come to the celebrated school of Plato, in which, during the life of its founder, Geometry formed the basis of instruction. It is delightful to think of the enthusiasm which so great a man as Plato felt for this study. He placed an inscription over the door of his school, saying, 'let no one who is ignorant of Geometry enter here.' He also declared to his disciples his belief, that *the mind of the Deity was constantly occupied with the truths of Geometry.* For some time the disciples of Plato shared the enthusiasm of their master, and accordingly from them Geometry received immense accessions. Leon, a pupil of one of Plato's disciples, arranged, for the second time, the elements of Geometry into a regular treatise. And Eudoxus, an intimate friend of Plato, found out *the solidity of a pyramid and cone.* It is also supposed that he was the inventor *of the theory of geometrical proportion,* as presented by Euclid, of whom we are next to speak.

About 300 years before Christ, Ptolemy Lagus founded a school of philosophy at Alexandria, in which Mathematics was cultivated before every thing else. It was here that Euclid gained his lasting celebrity as a Geometer; his ardor having been first kindled at Athens, under the disciples of Plato. It is related that when Ptolemy Philadelphus asked him, whether there was any easier method of studying Geometry than the one commonly pursued, he replied, 'No: there is no royal road to Geometry.' Euclid is chiefly known in modern times as the author of *The Elements*, a work composed with such wonderful judgment and sagacity, that the efforts of 2000 years have scarcely been able to make an improvement upon it. It has often been remodelled, and has had ever so many commentators; but under some form or other, it is at this day studied in every region of the civilized world. What a glorious earthly immortality did the composition of this work secure to its author! It is a singular fact that Euclid's Elements were first known to Europe, after the revival of learning in the 12th century, through the medium of an Arabic translation.

Following down the order of time, the next name of celebrity is that of Archimedes, who was born at Syracuse about 287 years before Christ. It was he who first discovered the properties of the *sphere* and *cylinder*. Upon these discoveries, he wished his fame with posterity to rest; for which reason, he requested that after his death, a sphere and cylinder might be inscribed on his tomb. But he made a great many other discoveries; and among the rest, that of *the approximate ratio of the circumference of the circle to its diameter*. He demonstrated that calling the diameter 1, the circumference is between $3\frac{10}{70}$ and $3\frac{10}{71}$; and the principles laid down by him in this demonstration, have formed the basis of all succeeding approximations. It is generally admitted that Archimedes holds the same rank among the ancients, as Newton and La Place among the moderns. The method of *Exhaustions*, described hereafter, was his invention.

About the time that Archimedes died, Apollonius was born, a man who acquired such reputation among his contemporaries, as to be familiarly known by the

name of the *Great Geometer.* His writings have fortunately been preserved, and together with those of Euclid and Archimedes, form the chief sources from which our knowledge of ancient Geometry is derived. After Apollonius, no very distinguished name occurs before the Christian Era.

With the Christian Era commences a long interval in which no brilliant discovery was made. Learning of every kind was now in the wane. Towards the end of the fourth century, two Mathematicians appeared, Theon and Pappus, who wrote some excellent commentaries upon former works, but produced nothing original. Hypatia, too, the illustrious daughter of Theon, and his successor in the chair of the Alexandrian school, was famed for her knowledge of Geometry, and for the sagacity displayed in her annotations upon Apollonius. But these are all who deserve to be mentioned even as commentators, for several centuries.

During the fifth, sixth, and seventh centuries, Geometry was chiefly cultivated by the Arabs and Persians. The Arabs, without contributing many new discoveries, translated most of the works of the Greek Geometers, and by thus preserving the lights of this branch of science from total extinction, made some remuneration to Europe for the general devastation which followed their inroads. The Persians were well acquainted with the Elements of Euclid, and made copious commentaries upon it. One of their most distinguished Geometers, Maimon-Reschid, conceived such a singular fondness for one of Euclid's propositions, that he wore the diagram for an ornament embroidered on his sleeve. The Persians call Geometry *the difficult science*, and have fantastic names for all the principal propositions. For example, they call the proposition respecting the square of the hypothenuse, *the bride*, and the converse of it *the bride's sister*.

Rome never had any distinguished Geometers. Cicero professed a high esteem for Mathematics, but did not write upon the subject. The Chinese have never cultivated Geometry to any great extent. When the Europeans first visited them, their knowledge extended little farther than the rules of Mensuration.

In Europe from the eighth to the thirteenth century Geometry with difficulty maintained a precarious exist-

ence. Here and there a solitary individual, in the retirement of a cloister, made it the subject of his contemplations. But on the whole, this period may be properly called the midnight of Geometry.

During the thirteenth and fourteenth centuries we begin to perceive the dawnings of a brighter day. Among the absurd opinions entertained in the dark ages, one of the most absurd was the belief in *Astrology*, or the influence exerted by the positions and motions of the heavenly bodies upon human affairs. Yet to this belief, more than to any thing else, we are indebted for the revival of Geometry. The vain attempt to foretell the destiny of an individual, *by casting his horoscope*, as it was called; that is, by ascertaining the relative positions of the planets at the time of his birth, led those who professed Astrology to an assiduous cultivation of Geometry, without which their calculations could not be made.

At length, however, as the darkness of ignorance and superstition began to be dissipated Geometry was studied from a nobler motive. Though it takes its name from the measurement of lands, yet its noblest application is to the spaces of the heavens. In other words, it forms the key to all our knowledge of Astronomy, by far the most sublime of sciences. With this view, Geometry began to be cultivated in the fourteenth and fifteenth centuries. At this time it numbered among its votaries, Wallingfort, the English poet Chaucer, Purbach, and Regiomontanus. But names now begin to thicken upon us in such numbers that we can only mention the most celebrated.

Cavalleri was born at Milan in 1598. He invented a new method of geometrical reasoning, called the method of *Indivisibles*. He considered a line as made up of an infinite number of points, a surface as made up of an infinite number of lines, and a solid as made up of an infinite number of surfaces. These infinitely small elements of the geometric magnitudes, he denominated *indivisibles*. The method of summing an infinite series of terms in arithmetical progression had long been known; and accordingly the process of comparing curves with straight lines, and measuring the area of surfaces, and the solidities of solids, was now rendered simple and summary. The method of indivisibles has

a decided advantage over the ancient method of *Exhaustions* ascribed to Archimedes, by being far less cumbrous and circuitous. To explain what is meant by the method of exhaustions, we will describe its application to a particular case. Suppose it were required to find the area of a circle. For this purpose, a polygon is inscribed in the circle, and another is circumscribed about it. Here then are two determinate areas, one less and the other greater than that of the circle. Thus two limits are fixed, within which the area sought must be contained; and these limits may be constantly brought nearer together, by increasing the sides of the two polygons. At length the difference between the two limits is reduced to a quantity too small to be estimated. It is then said to be *exhausted*, and the area of either of the polygons may be taken for the area of the circle. This is the method which Archimedes employed to find the ratio of the circumference to the diameter. It was also employed by Ludolph Van Ceulen, a Dutch Geometer contemporary with Cavalleri, for the same purpose. This man had the patience to carry the approximation to 36 figures.

Another contemporary of Cavalleri, Roberval of France, invented a method of reasoning which closely resembled the method of indivisibles; but differed in this, that surfaces were considered as made up of an indefinite number of narrow rectangles or oblongs, and solids of an indefinite number of thin prisms, all decreasing according to a certain law.

In the same century Descartes conferred a lasting benefit upon Geometry, by applying Algebra to it. By this invention, the properties of geometrical figures are represented by equations; and the *Application of Algebra to Geometry*, has now become an extensive branch of mathematics. The ancient Geometers were entirely ignorant of Algebra, and the discovery of so powerful an instrument, is the most important advantage yet gained by the moderns.

In this connexion we must not omit to mention Pascal, especially as we write for youth. Probably France never produced a greater genius. He had heard the mathematicians who visited his father speak with enthusiasm of Geometry. He requested that a book of Geometry might be given him. This his father refused,

because he was yet only twelve years old, and it was not consistent with the plan marked out for his education, that he should commence the study of mathematics so young. But Pascal was not to be thus put off. He had received a hint of what Geometry was, and immediately began to invent a system for himself. The walls of his room were literally covered with diagrams, and he had already advanced so far as to demonstrate, that *the three angles of a triangle are equal to two right angles*, when his father discovered what he was doing. This, be it remembered, was when he was only twelve years old.

Passing over Huygens and Gregory, we pause a moment to admire the enthusiasm of Dr Barrow, the illustrious preceptor of Newton. Though educated for a theologian, Geometry had attractions which he could not resist. He, like Plato, considered the contemplation of it, as not unworthy of the Deity, and inscribed the edition which he published of Apollonius, with these words: 'God himself geometrizes; O Lord, how great a Geometer thou art!'

It would seem that few discoveries now remained to be made in Geometry. The labors of the eighteenth century were chiefly directed to the extending of its applications, thus making it the instrument instead of the object of discovery. Or if any still attempted to improve the science itself, it was by remodelling its elements, and not by adding to their number. To this class belong Simson, Playfair, and Legendre. We might mention many others, but we limit ourselves to these, because they are the authors chiefly studied in the United States. Simson and Playfair, two Scottish professors, have each published improved editions of Euclid, which leave little to be desired on the subject of elementary Geometry, according to the ancient or Euclidean method. Legendre, the most eminent of French Geometers, has produced a work, which deservedly stands at the head of modern systems. It has been many times translated, and has passed through a great number of editions. The translation, which is chiefly studied in this country, was executed by Professor Farrar, of Harvard University. Respecting these three works, we shall only add, that those who would understand Geometry as it was left by Euclid, must study

Simson ; those who would unite modern improvements with the rigid method of the ancients, must study Playfair ; and those who would have a complete view of Geometry as it now is, without particular regard to the ancient method, must study Legendre.

As the student may desire to know in what respects the ancient and modern methods differ, we shall briefly state their general characteristics. Both agree in this, that certain principles or truths are taken for granted *to begin with*. They are taken for granted, because they cannot be proved ; being self-evident the moment they are stated. These are called *axioms*, and are to Geometry, what the foundations are to a building. Euclid's axioms are the following :

1. Things which are equal to the same thing are equal to one another.
2. If equal sbe added to equals, the wholes are equal.
3. If equals be taken from equals, the remainders are equal.
4. If equals be added to unequals, the wholes are unequal.
5. If equals be taken from unequals, the remainders are unequal.
6. Things which are double of the same, are equal to one another.
7. Things which are halves of the same, are equal to one another.
8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.
9. The whole is greater than its part.
10. Two straight lines cannot inclose a space.
11. All right angles are equal.
12. If a straight line meets two straight lines so as to make the two interior angles on the same side of it taken together, less than two right angles, these straight lines being continually produced, shall at length meet upon that side upon which are the angles which are less than two right angles.

The last of these has been added by Euclid's Commentators.

The two methods differ in this. Euclid never supposes a line to be drawn, until he has first demonstrated the possibility and pointed out the manner of drawing

it. But in three cases the possibility cannot be demonstrated, because it is self-evident. These cases are called *postulates*, and are the following:

1. Let it be granted that a straight line may be drawn from any one point to any other point.

2. Let it be granted that a terminated straight line may be produced to any length in a straight line.

3. Let it be granted that a circle may be described from any centre, at any distance from that centre.

The moderns, as Legendre, for example, are not thus scrupulous; but constantly suppose lines to be drawn, without demonstrating the possibility or explaining the manner.

Lastly, the two methods differ in this. The moderns avail themselves of all the aid which Algebra can afford them. The ancients were unacquainted with Algebra. Accordingly, Euclid was obliged to demonstrate the laws of proportion *geometrically*. Whereas in modern systems, these laws are supposed to have been previously demonstrated by the help of Algebra. The moderns also derive great advantage, in every part of Geometry, from the use of Algebraic signs and symbols. The ancient reasonings, for want of these, were rendered exceedingly cumbrous and circuitous.

These are some of the general distinctions. But the student who would be able to estimate the comparative merits of the two systems, must examine both for himself.

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ELEMENTS OF GEOMETRY.

SECTION FIRST.

OF LINES AND THEIR RELATIONS.

1. Definition.—*A point is position merely without any magnitude.* The study of Geometry properly begins with the consideration of *a point*, this being the first and simplest geometrical idea. If you were required to make a point with a pencil upon paper, you would merely place the sharpened end upon the paper, without moving it in any direction. If the pencil be as sharp as possible, this is the nearest approach you can make to a geometrical point as above defined. But as you cannot represent to the eye that which has absolutely no extension, it is sufficiently near the truth to call a point—*that which has an infinitely small extension.* By infinitely small, we mean for the present, the smallest that can possibly be conceived.

2. Def.—*A line is the path described by the motion of a point.* A point is the beginning and end of a *line;* for if you were required to make a line, you would begin by placing the point of your pencil upon the paper; you would proceed to move it along the surface of the paper; and you would end by ceasing to move it. Here you make one point by placing the pencil; you make a line by moving it; and you make another point where you cease to move it. These points are the boundaries of lines. It is evident that if the describing point had no extension, the line would only have that which it acquires from the motion, namely *length*, without any breadth or thickness. But as such a line could not be represented to the eye, it is sufficiently near the truth to say—*a line*

has length with only an infinitely small breadth and thickness.

3. Def.—*A straight line is the path described by a point moving only in one direction.* This will be readily understood if you consider how you would proceed to make a straight line. Your single endeavor would be to move the pencil throughout in one and the same direction. Thus if the pencil be placed at
F 1 A (fig. 1) and if it move only in one single direction till it reaches B, the line A B is a straight line.

4. Axiom.—*A straight line is the shortest way from one point to another.* By *axiom* is meant a proposition the truth of which is self-evident without reasoning. The above is one of this kind. If you were
F 1 standing at a point A (fig. 1), and were required to run to the point B in the shortest possible time, would you keep always in the straight line A B, or would you deviate from it ? You answer without a moment of hesitation, that you would keep in the straight line between the two points. Why? Because if you were to depart from it you would be obliged to return to it before you could reach B, since B is situated in it; and you would thus lose time. This is the only reason you could give; for if you were further asked why you would lose time by departing and returning, you could give no other reason for your belief than that the thing is self-evident, or no one can doubt it.

5. Scholium.—*How to make a straight line.* By *scholium* is meant any explanatory remark relating to what has gone before. If you ask how you can be sure, when attempting to make a straight line, that the describing point does not change its direction? We answer that in practice this assurance is obtained by moving the pencil along the edge of an instrument called a *rule*, which is already ascertained to be straight. The rule is ascertained to be straight, by *taking sight*, as it is called, upon its edge, it being a fundamental principle in optics that the rays of light move in straight lines.

6. Axiom.—*Two points determine the position of a*
F 10 *straight line.* If a single point be given as A (fig. 10), it is obvious that any number of straight lines may be drawn through it as in the figure, for the rule may be placed so as to have the point A coincide with its

edge, and may then be turned round so as to have ever so many different positions, the point A still coinciding with its edge. Therefore one point does not determine the position of a straight line. But if there be two points given as A and B (fig. 1) it is obvious F 1
that only one straight line can be drawn between or through them. Why? We might say because there can be but one shortest distance between two points. Or we might say because if the rule were so placed as to have the two points coincide with its edge, it could not be moved from this position without leaving one or both the points out of its edge. But neither of these reasons adds any force to our first belief. Hence the above is received as an axiom arising from the nature of a straight line.

7. Scholium.—*How to measure straight lines and express them by numbers—Linear Units.* Lines are measured, like all other quantities, by taking some known quantity of the same kind as a standard, and seeking how often it is contained in them. Thus the standard by which we measure a straight line, must be a straight line of a known length, as an inch, a foot, a yard, etc. This standard, whatever it be, is called a *linear unit*, and we have the measure of a straight line when we know the number of linear units it contains. Thus if we take an inch for the linear unit, and if we find it is contained 9 times in a given line as A B (fig. 1), we say the measure of A B is 9 F 1
inches. Since then the value of straight lines can be expressed in abstract numbers, and since abstract numbers are the object of arithmetic, it is obvious that the fundamental operations of arithmetic may be performed upon lines. This is called the application of arithmetic to geometry. Moreover, since algebra is nothing more than general arithmetic, it follows that algebra as well as arithmetic may be applied to Geometry.

8. Problem.—*To find the greatest common measure of two straight lines in order to express their ratio in numbers.* By *problem*, is meant an operation proposed to be performed. The performance of the operation is called the *solution* of the problem. We proceed to solve the above, taking A B and C D (fig. 2), F 2
for the two lines to be compared. The process is similar

to that in arithmetic for finding the greatest common divisor of two numbers. If the learner is not already acquainted with the use of signs, he must observe that the sign (=) signifies *equal to;* and the sign (+) which is called *plus*, signifies *added to.* SOLUTION.—First seek how often the smaller line C D is contained in the larger A B. If it were contained exactly 3 times for example, we should have their ratio at once, namely 3 to 1. That is A B would be 3 times as great as C D. But we find upon trial that C D is contained in A B twice and E B over. Therefore C D is not a common measure. We next apply the remainder E B to C D and find that it is contained once and F D over. Therefore E B is not a common measure. We next take the remainder F D and apply it to E B. It is contained once and G B over. Therefore F D is not the common measure. This process of applying the last remainder to the preceding must be continued as long as there is a remainder. If no such limit is attainable, the lines are said to be *incommensurable.* If this limit can be attained, the line last applied is the greatest common measure. Thus if G B is contained exactly twice in F D, G B is the common measure sought. The ratio is then expressed as follows. G B, the linear unit, is 1. Then, since it is contained twice in F D, F D=2. But E B=F D+G B=2+1=3. Again C D=E B+F D=3+2=5. Lastly A B=2 C D+E B=10+3=13. Accordingly the ratio of A B to C D is that of 13 to 5; that is A B is $\frac{13}{5}$ of C D, or C D is $\frac{5}{13}$ of A B.

9. DEF.—*A polygonal line is the path described by a point which changes its direction at intervals so large that they can be perceived.* If a line be not straight, it must be because the describing point has changed its direction once or more. When these changes of direction do not take place so often as to prevent your
F 3 perceiving the intervals A B, B C, C D (fig. 3) between any two successive changes, the line, which is made up of straight lines, is called a *broken* or *polygonal* line.

10. DEF.—*A curved line or curve is the path described by a point which changes its directions at intervals so small that they cannot be perceived.* When the direction changes so often that you cannot perceive the in-

tervals between the successive changes, as in A B (fig. 4,) the line thus described is called a *curved line.* F 4
In this case though you cannot actually perceive the intervals between which there is no change, yet this does not hinder your conceiving that there are such intervals. Indeed there must be such intervals X from the very nature of the motion, but the changes are so frequent that they are infinitely small. Corollary I.—*A curved line may be considered as made up of infinitely small straight lines.* By *corollary* is meant a proposition which follows as a consequence from a preceding proposition or definition. The above corollary follows directly from the definition of a curve, and furnishes the best idea you can form of it, for you thus make a straight line the *unit* or *element* of all lines, a principle which will be found to be of great utility hereafter, when we come to compare curved and sraight lines. Cor. II.—*A straight line and a curve can only coincide for an infinitely small extent.* To *coincide* is to lie one upon the other and exactly fill the same space. The above corollary follows directly from the two definitions of a straight line and a curve

THE CIRCLE AND ITS PARTS.

Def.—*A circle is a figure bounded by a curved line, all the points of which are equally distant from one point called the centre.* Thus if all the points in the curved line B C G D F E (fig. 5) are equally distant F 5
from the centre A, the figure is a circle. If the straight line A B, having the point A fixed, be supposed to turn about this point, till having performed a complete rotation, it returns to its first position, this motion would describe the circle. The moving line A B is called the *radius*, and the bounding line B C G D F E the *circumference.* Any straight line as D B drawn through the centre to meet the circumference each way is called a *diameter.* Any portion of the circumference as B C G is called an *arc.* Any straight line as G B joining the extremities of an arc is called a *chord.* Any portion of a circle comprehended between an arc and its chord is called a *segment*, as the

segment B G C. Any portion of the circle comprehended between two radii and an arc is called a *sector*, as the sector E A B. COR.—*In the same circle all radii are equal—each diameter is double the radius—all diameters are equal—every chord is less than its arc.* All these follow directly from the definitions.

12. THEOREM.—*Every diameter bisects the circle and its circumference.* By *theorem* is meant a proposition the truth of which is to be demonstrated by a process of reasoning. To *bisect* is to divide into two equal parts. To *enunciate* a proposition is to state it in words. We proceed to demonstrate the theorem above enunciated, by the method called *superposition.* DEMONSTRATION.—Let the two portions of the circle D E B,
F 5 D G B (fig. 5) above and below the diameter D B, be folded one upon the other, so that the folded edge shall coincide with the diameter. The two portions of the circumference will exactly coincide with each other; for if they did not, there would be points in them unequally distant from the centre, which would be contrary to the definition of a circle. Therefore the diameter bisects the circle and its circumference, which was to be demonstrated. Each portion of the circumference cut off by the diameter is called a *semicircumference*, and each portion of the circle is called a *semicircle*.

13. THEOREM.—*In the same circle or in equal circles, if two arcs are equal, their chords will be equal and conversely, if two chords are equal their arcs will be equal.* This is demonstrated in a manner similar to the preceding, that is by *superposition.* DEM. 1.—Let the
F 5 arc D F be supposed equal to D G (fig. 5). Then if the lower portion of the figure be folded upon the upper as before, the arcs D G and D F coinciding, the point G will fall upon the point F, and the chords D G and D F having two points D and F common must coincide throughout, since only one straight line can be drawn between two points (6). Therefore if the two arcs are equal, their chords are equal. 2.—We are now to prove the converse, namely if two chords as D G and D F are equal, their arcs are equal. If the chord D G be applied to D F, as they are by *hypothesis* or supposition equal, the point G must fall upon F. Then the arcs D G and D F, belonging to

the same circle and having two points common, must have all their points common, since they must all be equally distant from the centre by the definition. Therefore if two chords are equal their arcs are equal.

14. Prob.—*Having any arc given, to make another equal to it.* It is proper here to remark that the instrument used in making arcs and measuring them is called a *compass* or more generally *compasses*. Being very common we shall not describe it. Sol.—If the arc B C (fig. 6) be given and you wish to make another F 6 as D F equal to it, you first describe an indefinite arc D F with the same radius as that of B C, because they must belong to the same circle. Then take the chord B C in the compasses, and placing one foot in D move the other round till it cuts D F in F. The arc D F will be equal to B C because their their chords are equal (13.)

15. Scholium.—*How the circumference is divided.* It often becomes necessary to compare an arc with an entire circumference or with another arc of the same circumference. For this purpose every circumference is supposed to be divided into 360 equal arcs called *degrees* and marked thus (°). For instance 60° is read 60 degrees. As all circumferences whether great or small, are divided into the same number of parts, it follows that a degree, which is thus made the unit of arcs, is not a fixed value, but varies for every different circle. It merely expresses the ratio of an arc, namely $\frac{1}{360}$, to the whole circumference of which it is a part, and not to any other. As we sometimes have occasion for an unit less than a degree, each degree is divided into 60 equal parts called *minutes* and marked thus ($'$). Again each minute is divided into 60 equal parts called *seconds* and marked thus ($''$). When extreme minuteness is required the division is sometimes continued to *thirds* and *fourths*, marked thus ($'''$), ($''''$). As a quarter of a circumference, or, as it is generally called, a *quadrant* contains 90°, and as small numbers are more convenient than larger ones, it is usual in practice to refer all arcs to a quadrant, instead of an entire circumference. Thus considering a quadrant as unity, we say that a degree is $\frac{1}{90}$, a minute $\frac{1}{5400}$, and a second $\frac{1}{324000}$ of the quadrant in which it is taken.

ANGLES AND THEIR MEASURES.

16. Def.—*An angle is the opening made by two straight lines meeting or crossing each other.* Thus the
F 7 two lines A B, A C (fig. 7) which meet at A, form an opening B A C, which is called the angle B A C. The point of meeting A is called the *vertex* of the angle, and the straight lines A B, A C are called *sides.* The best way to obtain a definite idea of an angle is to suppose the line A B at first to coincide with A C, and then to turn about the fixed point A in the manner of a radius (11) till it reaches its present position. Then we may consider the angle as denoting—*the quantity by which a straight line, turning about one of its points, has departed from coincidence with another straight line.* To designate an angle, we make use of three letters as B A C, that at the vertex being in the middle. This is necessary when there are several
F 5 angles formed at the same vertex, as at A (fig. 5).
F 7 But if there be only one as at A (fig. 7), a single letter is sufficient to designate it.

17. Corollary.—*Angles are measured by means of the arcs of circles described from their vertices as centres.* This follows directly from the foregoing definition. It is obvious that the magnitude of angles does not depend at all upon the length of their sides, for
F 7 the angle A (fig. 7) is the same, according to the definition, whether we consider A D, A B, or A B produced, as the moving side. Now the measure of an angle must be some known magnitude which increases and diminishes simultaneously with the angle itself. Where shall we find such a magnitude? We answer the definition itself suggests one. For while the line A B moves as a radius about the fixed point A, every point in the line A B describes an arc of a circle; and since the arcs and the angle are formed by one and the same motion, beginning, increasing, and ending simultaneously, we have in the arcs thus formed every property included in the idea of a measure. Accordingly we express the measure of the angle at A by stating the number of degrees in the arc D F or B C, described from the vertex A as a centre. It is immaterial which we take, for each will contain the same

number of degrees (15) since each is the same fractional part of an entire circumference. If B C contain 15° then D F will contain 15°, and the angle at A will be an angle of 15°.

18. PROBLEM.—*Having any angle given to make another angle equal to it.* Let the given angle be A (fig. 8,) and let the vertex of the required angle be D. SOLUTION.—Draw the straight line D F indefinitely. Then with the centre A and any convenient radius, describe the arc B C. Again with the centre D and the same radius, describe the arc F E and make it equal to B C, by taking B C in the compasses, placing one foot at F and making the point E where the other meets the arc (14). We have now two points D and E, through which the remaining side of the required angle is to pass. These determine its position (6). Draw D E, and the angle D will be equal to A; for the arcs are equal because their chords if drawn would be equal (13); and the angles are equal because the arcs which measure them are equal (17). SCHOL.—This problem, as well as many others of similar nature, is more readily solved in practice, by means of a small metallic semicircle called a *protractor*, which is accurately graduated, that is, divided into degrees, and which is usually found in cases of mathematical instruments. If, for example, we wish to make an angle D (fig. 8) equal to 40°, we apply the diameter of the protractor to the straight line D F, so as to make the notch marking the centre fall exactly at the point D intended for the vertex; then we have only to seek the number of 40, mark the point, and draw the other side through the vertex and this point. F 8

19. DEF.—*A right angle is that which is made by one straight line meeting another, so as to make the two adjacent angles equal. An Acute angle is less than a Right angle. An Obtuse angle is greater than a Right angle.* Thus when the moving line A B (fig. 9,) has reached that position in which the two adjacent angles B A C and B A D are equal to each other, these are called *right angles*. In this case A B is said to be *perpendicular* to C D; so that to say a line is perpendicular to another, and to say a line makes a right angle with another, are the same thing. If the moving line has not reached the position of A B, the angle is called an *acute angle*, as E A C. If it has passed beyond F 9

A B, the angle is called an *obtuse angle*, as F A C. In each case, one of the lines is said to be *oblique* with respect to the other. Thus A E and A F are oblique with respect to A C.

20. Cor.—*A right angle has for its measure a quadrant or ninety degrees.—All right angles are equal.—All the angles which can be formed about one point, are equal to four right angles.* All these follow directly from the preceding definitions. Thus, since D A B
F 10 and D A C (fig. 10), are equal by the definition of right angles, and since both are measured by the semicircumference B D C or 180°, it follows that each of the right angles is measured by half this quantity, that is, a quadrant or 90°. Again, all the right angles B A D, D A C, C A E, E A B, are equal, because all have the same measure, namely, a quadrant. Lastly, all the angles which can be formed at the point A are equal to 4 right angles, because they have for their measure an entire circumference or 360°.

21. Def.—*The complement of an angle is the difference between that angle and a right angle. The supplement of an angle is the difference between that angle and two right angles.* Cor.—*Equal angles have equal complements and equal supplements, and the converse.*
F 9 Thus E A C (fig. 9), is the complement of E A B, and the supplement of E A D. If E A C is 30°, its complement is 60° and its supplement 150°. The corollary is evident, because it is an axiom that if equals be taken from equals, the remainders are equal.

22. Theorem.—*When one straight line crosses another, the two adjacent angles are supplements of each other, and the two vertical angles are equal to each other.* By *adjacent angles* is meant those which are *next* to each other, as C E B and B E D, or C E A and
F 11 A E D, (fig. 11.) By *vertical* angles is meant those which are opposite to each other, as C E A and B E D, or C E B and A E D. Dem. 1.—C E B and B E D are supplements of each other, because if a circle were described with E as a centre, these two angles would be measured by a semicircumference. Therefore they are equal to two right angles. And C E B is 180° minus B E D; also B E D is 180° minus C E B. The same would be true of C E A and

A E D. 2. Again we say that C E A is equal to B E D, because from what has just been proved, C E B is the supplement of both. And C E B is equal to A E D, because B E D is the supplement of both.

23. Theorem.—*If a perpendicular be erected upon the middle of a straight line, every point in the perpendicular is equally distant from the extremities of that line.* Let A B (fig. 12,) be the line, D the middle of it, and C D the perpendicular. We are to prove that every point in C D is equally distant from A and B. Dem.—In the first place D is equally distant by *hypothesis*, that is by the conditions of the proposition. Now take any other point at pleasure as C, and draw C A and C B. We say C A=C B. For let the figure C D A be folded upon C D B so that the folded edge shall coincide with C D. Then since the angle C D A=C D B, being right angles, the line D A will fall upon D B; and since they are by hypothesis of the same length, the point A will fall upon B. Therefore, since C A and C B have the two points C and B common, they must coincide (6) and are equal. C then is equally distant from A and B; and since C was taken *at pleasure*, that is, any where in the perpendicular, the same is true of every other point, and the proposition is demonstrated. F 12

24. Problem.—*To erect a perpendicular at a given point in a straight line.* Sol.—Let A B (fig. 13) be the given line and C the given point. Place one foot of the compasses in C and fix two points A and B at equal distances from C. Then with A as a centre and any radius greater than A C, make an arc D, and with B as a centre and the same radius, make another arc cutting the first in D. The point D thus fixed, is equally distant from A and B; because both the arcs are described with the same radius from the centres A and B. The point C was made so at first. Therefore the line D C having all its points equally distant from A and B, by the preceding proposition, is the perpendicular required; this being evidently the property of no line but a perpendicular. F 13

25. Problem.—*From a given point without a straight line, to let fall a perpendicular to that line.* Let A (fig. 14,) be the given point and B C the given line. Sol.—With A as a centre and any radius greater F 14

than the shortest distance from A to the line B C, make an arc cutting B C in two points B and C. A is equally distant from these two points. Then find another point D, as in the preceding problem, which is equally distant from B and C. The straight line drawn through A and D (23) is the perpendicular required.

26. PROBLEM.—*To bisect a given straight line, or*
F 15 *to find the middle of it.* Let A B (fig. 15,) be the given line. SOL.—By the method before described (24) fix two points C and D at equal distances from the extremities A and B. Then draw the straight line C D and it will be perpendicular to the middle of A B (23.) Therefore the point E is in the middle of A B. SCHOL.—This is conveniently done in practice, by taking a string equal to A B, then doubling it, and measuring from A or B. It will reach to the middle.

27. PROBLEM.—*To bisect a given arc or angle.*
F 16 Let A (fig. 16) be this angle. SOL.—With the centre A and any radius, make an arc B C to measure this angle. Draw the chord B C. A is equally distant from B and C. Fix another point D equally distant from B and C, and draw A D. A D is perpendicular to the middle of B C (23.) Therefore E, one of its points is equally distant from B and C, and the chord E B=the chord E.C. Then (13) the arc B E=the arc E C, that is, the arc B C is bisected at E. Also the angle A is bisected, for B A E=E A C, being measured by the equal arcs B E and E C.

28. THEOREM.—*If a perpendicular be erected upon the middle of a chord, it will pass through the middle of the arc, and through the centre of the circle to which the arc belongs.* DEM.—The truth of this proposition is evident from the preceding construction. A D
F 16 (fig. 16,) is perpendicular to the middle of B C, it passes through the centre A, and it bisects the arc B C. It is moreover evident that no line can be perpendicular to the middle of B C which does not pass through A; for if there were such a line, it would differ from A F, and then the angle which it would make with F C, would be either greater or less than a right angle, which would contradict the supposition of its being perpendicular. Hence the proposition enunciated must be universally true.

29. PROBLEM.—*To find the centre of a given arc or circumference, or of one which shall pass through any three points not in the same straight line.* SOL. 1.—Let the circumference A D C B (fig. 17,) or any portion of it be given. In order to find the centre, take any three points A, D, C and join them by the chords A D, D C. Erect a perpendicular upon the middle of A D and it will pass through the centre (28.) Erect another upon the middle of D C and it must also pass through the centre. But a point which is in two lines at once must be at their intersection. Therefore E is the centre sought. 2.—If we had any three points given, as A, D, C, not in the same straight line, the process for finding the centre of the circumference passing through them, would be the same, for the point E thus found, is equally distant from the three points in question (23,) and therefore the circumference described with the centre E so as to pass through one, must pass through the other two. Moreover as the above construction is independent of any particular situation of the points A, D, C, it is possible so long as these three points can be joined by two straight lines, which can be made two chords of a circle, since all that is necessary is to bisect these two chords by perpendiculars. But if the three points were in the same straight line, there would be no longer two straight lines joining them, but only one, and the construction would be manifestly impossible. F 17

30. THEOREM.—*A perpendicular measures the shortest distance of a point from a line.* Let A (fig. 18) be the point in question, and B E the line. We say that the perpendicular A D is shorter than any oblique line as A C. This, as well as the following proposition, is very nearly self-evident, but it is usual to give a demonstration of them. DEM.—With A as a centre and a radius A C, make the arc C F, and produce it till it cuts C E in E. Now it is evident from the second corollary in article 10th, that the chord C E can never coincide with the arc C F E, so long as C E is of any perceptible magnitude, that is, so long as C is taken at any appreciable distance from D. But so long as these do not coincide, A D will be less than A F, and therefore less than its equal A C. Accordingly the perpen- F 18

dicular. A D is shorter than any oblique line A C, however near to D the point C be taken.

31. THEOREM.—*Two oblique lines drawn equally distant from the perpendicular are equal, and of two oblique lines drawn unequally distant, the more remote is*
F 18 *the greater.* DEM. I.—First let A C and A E (fig. 18) be drawn at equal distances from A D. We say they are equal. By the supposition A D is perpendicular to the middle of C E. Therefore (23) the point A is equally distant from C and E; in other words A C= A E which was first to be demonstrated. II.—Again let A B be more remote from the perpendicular than A C. We say that A B is greater than A C. Let A B be supposed to turn about A as a centre till it coincides in direction with A C. The point B will describe the arc B G, and this arc will always differ from the straight line B C so long as B C is of any perceptible magnitude. Accordingly A C will be less than A G, and therefore less than its equal A B. Or, as the proposition was enunciated, A B will be greater than A C, so long as it is more remote, by however small a quantity, from A D.

32. THEOREM.—*There can be only one perpendicular let fall from a point to a straight line, and there can be only one perpendicular erected at a point in a line.*
DEM.—The first part of this proposition is true, because there can be but one *shortest* distance from a point to a line. The second part is true, because all right angles are equal, which would not be the case if all perpendiculars erected at the same point did not coincide and actually form but one perpendicular.

PARALLEL LINES.

33. DEF.—*Two straight lines are parallel when all the perpendiculars let fall from points in one to the other are equal.* When two straight lines as A B and
F 19 C D (fig. 19,) are so drawn as to be throughout at the same distance from each other, they are said to be *parallel.* But we have already shown (30) that the shortest distance of a point from a line is measured by a perpendicular. Thus the shortest distance of the point L from the line C D is the perpendicular L M, and the

same is true with respect to any other point. Accordingly we assume the above definition as the basis of our reasoning upon parallel lines. COR.—*Two parallel straight lines can never meet each other, however far produced*, for if they have any distance at first, they must always have it.

34. THEOREM.—*When two parallels are crossed by a straight line, the alternate-internal angles are equal to each other, and the internal-external angles are equal to each other.* If two parallel straight lines, as A B and C D (fig. 20), are cut by a third straight line E F, the angles A H G and H G D, when named together, are called *alternate-internal angles*, because they are on opposite sides of the single line E F and within the parallels A B and CD. For the same reason B H G and H G C are *alternate-internal angles*. Again the angles F H B and H G D are called *internal-external angles*, because one is within and the other without the parallels, and both are on the same side of the single line. For the same reason A H G and C G E are *internal-external angles*. DEM. 1.—We are to prove that A H G is equal to H G D. From H let fall the perpendicular H L to C D. From G let fall the perpendicular G M to A B. By the definition (33) G M=H L. Produce H L making L K=H L. Produce G M making M I=G M. Then H K=G I. Moreover C D is perpendicular to the middle of H K, and A B is perpendicular to the middle of G I. Therefore if with the centre H and radius H G an arc G O I be descibed, this arc will be bisected at O (28). Also if with the centre G and the same radius G H, the arc H N K be described, this arc will be bisected at N. Now as the chord H K=the chord G I, the arc H N K=the arc G O I (13;) and since these are bisected at N and O, it follows that the arc H N=the arc G O. But the arc HN measures the angle H G D, and the arc G O measures the angle A H G. Therefore A H G=H G D, which was the first thing to be demonstrated. 2.—We are to prove that F H B=H G D. Now F H B=A H G because they are vertical (22). But we bave just proved that A H G=H G D. Consequently F H B=H G D; each being equal to A H G; for it is an axiom that things which are equal to the same thing are equal to each other.

35. THEOREM.—*When two parallels are crossed by a straight line, the sum of the two interior angles on the same side is equal to two right angles.* If two parallel
F 21 straight lines A B and C D (fig. 21) meet a third straight line G H, the two angles A I K and C K I are called *interior on the same side*, because they are within the parallels and on the same side of the single line. For the same reason L I K and I K M are called interior on the same side We are to prove that A I K+I K C=2 right angles. DEM.—A I G+A I K=2 right angles, because they are supplements of each other (22). But A I G=I K C being internal-external angles (34). Therefore, substituting I K C in the place of its equal A I G, we have A I K+I K C=2 right angles, which was to be demonstrated.

36. THEOREM.—*If a straight line is perpendicular to one of two parallels, it is also perpendicular to the other, and if two lines are perpendicular to a third they are parallel.* DEM. 1.—First we say that if E F
F 21 is perpendicular to A B (fig. 21) it is also perpendicular to C D. We here take it for granted that A L M is a right angle, and we are to show that L M C is also a right angle. Now A L M+L M C=2 right angles, being interior on the same side (35). Then if from two right angles we take one right angle A L M, there must remain a right angle L M C. Hence E F is perpendicular to C D, which was the first thing to be demonstrated. 2.—The second part hardly needs demonstration, but it can be demonstrated as follows. If A B and C D are perpendicular to E F, we say they are parallel. For if A B is not parallel to C D, there can be a line drawn through L different from A B, which shall be parallel to C D. But then if it differ from A B, it cannot make an angle A L M, which added to L M C, shall make the two interior angles on the same side equal to two right angles (35). Therefore no line different from A B can be parallel to C D, and A B itself must be parallel to C D.

37. PROBLEM.—*Through a given point to draw a straight line parallel to a given straight line.* Let
F 22 A B be the given line and C the given point (fig. 22). SOL.—With C as a centre and any convenient radius as C D, make an indefinite arc D F. With D as a centre and the same radius make the arc C G. Then make

the arc D F=C G (14). Through C and the point F thus determined, draw C F and it will be the parallel required. For if C D be drawn, the angle A D C =the angle D C F, because their arcs are equal (18). Now we have proved (34) that if A B and C F were parallel, A D C would be equal to D C F being alternate-internal; and since no line different from C F could make D C F=A D C, we conclude that C F, which fulfils this condition, must be the parallel required.

38. THEOREM.—*Two parallels comprehended between two other parallels are equal.* Let A B and C D (fig. 23) be two parallels and E F and G H two parallels drawn between them. We wish to prove that E F=G H. DEM.—If they are perpendicular to the other two, this is evident from the definition (33). But suppose they are oblique. Still we say that E F= G H For since F I=H B by definition, F I may be placed upon H B. Then since E I F=G B H being right angles, the point E must fall somewhere in G B. Again since E F I=G H B being complements of the internal-external angles E F C and G H F, the point E must fall somewhere in H G. Now since E is to be at the same time in B G and H G, it must be at their intersection G. Hence E F=G H. SCHOL.—This proposition explains the nature of an instrument, by which parallel lines are drawn with much greater facility than by the process described in the preceding article. Let A B and C D represent two parallel pieces of wood, and E F and G H two parallel cross-pieces. The cross-pieces are connected with the parallels by pivots at each of their extremities E, F, G, H. Then by varying the obliquity of the cross-pieces, the distance between the two parellels may be varied at pleasure, without destroying their parallelism. F 23

39. THEOREM.—*Two angles which have their sides parallel and directed the same way, are equal.*—Let the two angles be E D F and B A C (fig. 24). DEM.—Produce E D to G. Since B A and E G are parallel, the angle B A C=E G C (34). Again since D F and A C are parallel, the angle E D F=E G C. Therefore B A C=E D F, each being equal to E G C. F 24

40. THEOREM.—*Two parallels whether tangents or secants, intercept upon the circumference equal arcs.*

F 25 If a line E F (fig. 25) touches the circumference of a circle only in one point I, it is called a *tangent*. We draw a tangent to any point, by making it perpendicular to the extremity of the radius at that point. For since every line G K drawn to a point different from I, would be an oblique line, and therefore greater than the radius G I, it follows that I is the only point common to the straight line and the curve. But if a straight line cuts the circumference in two points, as C D or A B, it is then called a *secant*. We observe that the sign (—) *minus* means that the quantity before which it is placed, is *subtracted*. DEM. 1.—We shall first take the case of two secants. We say then that the arc L M=N O. For bisect the chord M O by the perpendicular G H, and you bisect also its arc M H O (28). Therefore M H=H O. Again G H is also perpendicular to L N (36), and passes through the middle of it; for if it did not, a perpendicular might be erected at the middle of L N, which would pass through the centre G (28), and then we should have two perpendiculars drawn from the same point to the same straight line, which is impossible (32). Therefore G H is perpendicular to the middle of the chord L N and bisects its arc, so that L H=H N. Hence M H—L H=H O—H N or L M=N O. 2. Take now the case where one of the parallels is a tangent and the other a secant, as C D and E F. We say that M I=O I; for they are what remain after taking from the equal semicircumferences H M I and H O I, the equal arcs H M and H O. Lastly, if both the parallels were tangents, the arcs in question would be semicircumferences, and therefore equal.

41. THEOREM.—*An angle which has its vertex in the circumference of a circle, has for its measure half the arc intercepted between its sides.* If an angle has its vertex in the circumference, it must either be form-
F 26 ed by a tangent and a chord, as B A C (fig. 26), or it must be formed by two chords, as B A I. We shall demonstrate that the proposition above enunciated is true in both cases. DEM. 1.—First we say that the angle B A C formed by a tangent and a chord, has for its measure half the arc B G A. Draw the diameter D E parallel to B A, and the diameter F G parallel to A C. The angle B A C=D H G (39). Therefore

B A C has for its measure an arc equal to D G. It only remains, then, to prove that D G=half of B G A the intercepted arc. Now if D G be taken from B G A, we have remaining B D+A G. But B D+A G=F E, since B D=A E (40), A G=A F, and A E+A F =F E. Moreover F E=D G, since they measure vertical angles. Therefore B D+A G=D G, and D G is half of B G A. But D G measures B A C. In like manner I A C is measured by half the arc I G A. Therefore B A C is measured by half of B G A, that is, by half of the arc comprehended between its sides. 2. We are next to show that B A I, formed by two chords, has for its measure half of B I. Now B A I is the difference between I A C and B A C, and must therefore have for its measure the difference between their measures, that is, the difference between half of I G A and half of B G A, which is half of B I.

42. COROLLARY.—*All angles inscribed in the same segment are equal, and all angles inscribed in a semicircumference are right angles.* If an angle be formed by two chords, as A C B (fig. 27), it is called an *inscribed angle*. Then this corollary follows directly from the preceding proposition. Thus the angles A C B, A D B, A G B are equal, because they have for their measure half the arc A F B. Also A B F and A E F are equal, being measured by half of A F. Again, A B being a diameter, the angle A C B, A D B, A G B are right angles, because their measure is half the semicircumference A F B, that is, a quadrant. F 27

43. THEOREM.—*An angle whose vertex is between the centre and the circumference, has for its measure half the intercepted arc, plus half the arc contained between its sides produced;—and an angle whose vertex is without the circumference, has for its measure half the concave arc intercepted between its sides minus half the convex arc.* DEM. 1.—First we are to prove that B A C (fig. 28) has for its measure half of B C+half of H I. Produce B A and C A, and draw K H parallel to B I. Then B A C=K H C (39.) But K H C has for its measure (41) half of (K B+B C). Now since K B=H I (40), we have half of (K B+B C)=half of (H I+B C), which was to be proved. 2.—In the second place, we are to prove that C D E has for its measure half of C E—half of F 28

L F, or half of (C E—L F). Draw G F parallel to C D. Then C D E=G F E (39). But G F E is measured by half of G E. Now since G E=C E—C G=C E—L F, half of G E=half of (C E—L F), which was to be proved.

TRIANGLES.

44. Def.—*A triangle is a figure bounded by three straight lines.—An equilateral triangle is one which has its three sides equal.—An isosceles triangle is one which has two sides equal.—A scalene triangle is one which has no two sides equal.—A right triangle is one which has one right angle.* Scho.—A triangle is the simplest figure that can be bounded by straight lines. It is important that the properties of triangles be well understood, because as we shall see hereafter, all figures bounded by straight lines may either be divided into several triangles, or reduced to one equivalent triangle.

45. Theorem.—*Every triangle may be inscribed in a circle.* A triangle is said to be inscribed in a circle when it has its three vertices in the circumference as
F 30 A B C (fig. 30). Dem.—We have already shown (29) in what manner the circumference of a circle may be made to pass through any three points not in the same straight line. Therefore, since the three vertices of a triangle can never be in the same straight line, it follows that the circumference of a circle may be made to pass through them. The triangle will then be inscribed. Thus every triangle can be inscribed in a circle.

46. Theorem.—*The sum of the three angles of a triangle is always equal to two right angles.* Dem.—The
F 30 triangle A B C (fig. 30) being inscribed, each of its angles is measured by half the arc contained between its sides (41.) Thus A is measured by half the arc B C, B by half the arc A C, and C by half the arc A B. But these three arcs make up the whole circumference. Therefore the three angles have for their measure a semicircumference. Hence they must be equal to two right angles.

47. Corollary.—*No triangle can have more than one right angle or one obtuse angle; and the two acute angles of a right triangle are complements of each other.* This follows directly from the preceding. For first, if a triangle could have two right angles, or

two obtuse angles, there could be no third angle, and the figure would not be a triangle And secondly, in the right triangle B A C (fig. 34) since the three angles A, B, C are equal to 2 right angles, and since A is a right angle, the two acute angles B and C are together equal to a right angle. Therefore each is a complement of the other. F 34

PROBLEM.—*Two angles of a triangle being given, to find the third angle.* SOL.—This may be done *arithmetically*, by adding the degrees in the given angles and then subtracting their sum from 180°; or it may be done *geometrically*, by taking a semicircumference and cutting off two arcs equal to those which are used to measure the given angles. Then the remaining arc will be the measure of the angle required. This is very readily done by means of a protractor (18). COR.—*If two angles of one triangle are equal to two angles of another triangle, the remaining angles are equal.*

49. THEOREM.—*The exterior angle is equal to the sum of the two opposite interior angles.* By *exterior* angle is meant the outward angle formed by producing one of the sides of a triangle. Thus in the triangle A B C (fig. 29), B C D is the exterior angle, and A and B are the two opposite interior angles. We say that B C D=A+B. DEM.—For if B C D be taken from 180°, B C A will remain. Also if A+B be taken from 180°, B C A will remain. Now those things which, when taken from the same thing, leave equal remainders, must themselves be equal. Therefore B C D=A+B. F 29

50. THEOREM.—*If a triangle is isosceles, the angles opposite to the equal sides are equal and the converse.* If the side A B=the side A C (fig. 30,) we say the angle B=the angle C. DEM.—For since the chords A B and A C are equal, the arcs A B and A C are equal. Then half the arc A B=half the arc A C. But these measure the angles B and C. Therefore the angle B=the angle C. By similar reasoning we prove the converse of this proposition, namely—*if two angles of a triangle are equal, the triangle is isosceles.* For if the angles B and C are equal, the arcs A B and A C are equal. Then the chords A B and A C are equal, and the triangle is isosceles. F 30

51. THEOREM.—*If a triangle is equilateral it is*

equiangular and the converse. DEM.—If the three
F 30 chords are equal (fig. 30) the three arcs are equal. Then their halves, which measure the three angles must be equal. Consequently the angles themselves must be equal. Conversely—*if a triangle is equiangular it is equilateral.* If the triangle A B C is equiangular, the three arcs are equal. Then the three chords must be equal, and the triangle is equilateral.

52.—THEOREM. *In any triangle the greater side is opposite to the greater angle and the converse.* If the
F 31 angle B (fig. 31,) is greater than A, we say that the side A C is greater than the side B C. DEM.—For in this case the arc A C must be greater than the arc B C, since half of A C measures a greater angle than half of B C. But then the chord A C must be greater than the chord B C. Conversely—*in any triangle the greater angle is opposite to the greater side.* For if the chord A C exceeds the chord B C, the arc A C must exceed the arc B C. Then half of A C exceeds half of B C. Consequently B exceeds A, which was to be proved.

53. THEOREM.—*Two triangles are equal, when they have two sides and their included angle respective-*
F 32 *ly equal.* If A B (fig. 32)=D E, A C=D F, and the included angle B A C=the included angle E D F, then we say the triangles A B C and D E F are equal in all their parts. This may be proved by superposition. DEM.—Place D E upon A B, and by hypothesis they must coincide. Also, since the angle A=D, D F will take the same direction as A C, and since they are equal in length, the point F will fall on C, as E did on B. Then E F and B C, having two points common, cannot differ. The two triangles, therefore, coincide throughout. COR.—*Two sides and their included angle determine the triangle;* for while these three parts do not vary, the other three, namely, the remaining side and the two remaining angles, cannot vary.

54. PROBLEM.—*Having two sides of a triangle and their included angle given, to construct the triangle.* SOL.—Draw D F=to one of the given sides
F 32 (fig. 32). Make the angle D=to the given angle. This determines the direction of D E, and its length, which is given, determines the point E. Thus we have two points E and F, which determine the length and

position of the side E F. The triangle is therefore constructed. For, by the preceding proposition, there can be no triangle different from D E F, which has the same three parts given.

55. THEOREM.—*Two triangles are equal, when they have a side and two adjacent angles respectively equal.* If the side A B=D E (fig. 32,) the angle A=D, and the angle B=E, then we say the two triangles are equal in all their parts. The proof is by superposition as before. DEM.—A B will coincide with D E by hypothesis. B C will take the same direction as E F, because the angle B=E, and therefore the point C must fall somewhere in E F. Again A C will take the same direction as D F, because the angle A=D, and therefore the point C must fall somewhere in D F. Now since C is to be in both the lines D F and E F at the same time, it can only be at their intersection F. Thus the two triangles coincide throughout. COR.—*A side and its two adjacent angles determine the triangle.* F 32

56. PROBLEM.—*Having a side and its two adjacent angles given, to construct the triangle.* SOL.—Draw A B (fig. 32) equal to the given side. At A make an angle equal to one of the given angles. This will determine the direction of A C. At B make an angle equal to the other given angle. This will determine the direction of B C. The meeting of A C and B C determines the triangle, since, by the preceding proposition, no triangle having the same parts given can differ from the one constructed. F 32

57. THEOREM.—*Two triangles are equal, when their three sides are respectively equal.* If A B=D E (fig. 33), A C=D F, and B C=E F, then we say the two triangles are equal. DEM.—Place A C upon D F. Then it is only necessary to prove that the point B will fall upon E. Take D as a centre and with a radius D E make an arc at E. B must fall somewhere in this arc, because A B=D E. Again take F as a centre and with a radius F E make another arc cutting the first. The point B must fall somewhere in this arc also, because B C=E F. Then it can only be at their intersection E, and the triangles must coincide throughout. COR.—*Three sides determine the triangle.* F 33

58. PROBLEM.—*Having three sides of a triangle given, to construct the triangle.* SOL.—Draw D F

F 33 (fig. 33) equal to one of the given sides. Take D as a centre and with a radius equal to another of the given sides, make an arc E. Again take F as a centre and with a radius equal to the remaining side, make another arc cutting the first. Then draw D E and F E, and the triangle is constructed, since, by the preceding proposition, no triangle having the same things given, can differ from the one in question.

59. THEOREM.—*Two right triangles are equal, when they have the hypothenuse and another side equal.* By *hypothenuse* is meant the side opposite to the right angle. Thus B C is a hypothenuse and E F is a
F 34 hypothenuse. If B C=E F (fig. 34), and A C=D F, then we say the two right triangles are equal. DEM.—Place A C upon its equal D F. Then A B will take the direction of D E, because A and D are right angles, and the point B will fall somewhere in D E. We wish to prove that it will fall on E. Take F as a centre and with a radius E F make an arc cutting D E in E. B must fall somewhere in this arc, because B C=E F. Now since B must be at the same time in the line D E and in the arc, it can only be at their intersection E, and the two triangles coincide throughout. COR.—*The hypothenuse and a side determine a right triangle.*

60. PROBLEM.—*Having the hypothenuse and another side given, to construct a right triangle.* SOL.—Draw D F equal to the given side, and erect a perpendicular at D. Then take F as a centre and with a radius equal to the given hypothenuse, make an arc E cutting the perpendicular. Draw F E and the triangle is constructed, since, by the preceding proposition, no right triangle having the same things given can differ from the one in question.

61. COROLLARY.—*In order to construct a triangle or convince ourselves of its equality to another triangle, we must always know three of its six parts, of which one at least must be a side.* This follows from the eight preceding propositions. Three angles alone are not sufficient to determine a triangle. Because any number of different triangles may be constructed, all having their three angles respectively equal. Thus the
F 35 triangles A B C and D E F, (fig. 35) having their sides parallel, are equiangular with respect to each oth-

er (39). That is, A=D, B=E, and C=F. Yet the triangles are not equal; and it is evident that the number might be increased to any extent, and the same would be true.

OF PROPORTIONS.

62. Def.—*A proportion expresses the equality of two ratios.* We have already shown (8) that the ratio of two straight lines is expressed in the same manner and has the same meaning, as that of two abstract numbers. We now remark that—*two equal ratios in lines as well as in numbers, make a proportion.* Hence the phrase, *geometrical proportion.* To explain the nature and laws of proportion belongs to arithmetic and algebra. We shall not therefore enter into a particular analysis of them here. But for the sake of those who may not have studied proportion elsewhere, we shall briefly state the principles to be made use of hereafter. We shall illustrate the application of each by one example in numbers, since we have already shown that the value of straight lines may be represented by numbers (7). The ratio between two numbers is expressed in the form of a fraction. Thus the ratio of 6 to 9 is $\frac{6}{9}$. A proportion expresses the equality of two ratios. Thus the equation $\frac{6}{9}=\frac{10}{15}$ is a proportion. But the usual form of writing it is 6 : 9 : : 10 : 15. This is read, 6 is to 9 as 10 is to 15, and the meaning is, that 6 is the same part of 9 that 10 is to fifteen. The first term in each ratio is called an *antecedent*, and the second a *consequent.* Thus 6 and 10 are antecedents, and 9 and 15 consequents. The first and fourth term of a proportion are called *extremes*, and the second and third *means.* Thus 6 and 15 are extremes, and 9 and 10 means. If the same number is taken twice as a mean, it is called a *mean proportional.* Thus in the proportion 2 : 4 : : 4 : 8, we say 4 is a mean proportional between 2 and 8. If more than two equal ratios are written after one another, they form a *continued proportion.* Thus 6 : 9 : : 10 : 15 : : 8 : 12 is a continued proportion.

63. *In every proportion, the product of the means is equal to the product of the extremes.* For if two

equal fractions be reduced to a common denominator, their numerators must be equal. Thus from the proportion 6 : 9 : : 10 : 15 we have $9\times10=6\times15$. This property being universal, furnishes a convenient *test* by which to ascertain the truth of a proportion, for any four numbers will be in proportion when they satisfy this condition.

64. *If two proportions have one ratio common, the other two ratios make a proportion.* For ratios are nothing more than fractions, and two fractions, each of which is equal to a third, are equal to each other. Therefore these two make a proportion. Thus if we have 6 : 9 : : 10 : 15 and 6 : 9 : : 16 : 24, then we say 10 : 15 : : 16 : 24. Apply the test and this last proportion will be found true.

65. *In every proportion the means, or the extremes, or both, may change places.* For this does not affect the equality of the product of the means to that of the extremes. Thus the proportion 6 : 9 : : 10 : 15 may be written in the three following forms; 6 : 10 : : 9 : 15; 15 : 9 : 10 : : 6; 15 : 10 : : 9 : 6. Apply the test and all these will be found true.

66. *In every proportion, either ratio or both ratios may be multiplied or divided by the same number, without destroying the proportion.* For ratios are fractions, and to multiply or divide the numerator and denominator of a fraction by the same number does not alter its value. Thus from the proportions 6 : 9 : : 10 : 15 we have $\frac{6}{3}$: $\frac{9}{3}$: : 10×2 : 15×2. Apply the test and this last proportion will be found true.

67. *Every proportion may be multiplied by itself or by another, term by term, and the squares or products will form a new proportion.* For if two equal fractions be multiplied by two equal fractions, the products must evidently be equal fractions, that is equal ratios, and therefore a proportion. Thus if we have 2 : 4 : : 6 : 12 then we say 2×2 : 4×4 : : 6×6 : 12×12. Again if we have the two proportion 2 : 4 : : 6 : 12 and 6 : 9 : : 10 : 15 then, multiplying term by term, we say 2×6 : 4×9 : : 6×10 : 12×15. Apply the test and both will be found true.

68. *In every proportion, the sum of the two first*

terms is to that of the two last, and the difference of the two first is to that of the two last, as the first is to the third, or as the second is to the fourth. Thus from the proportion 2 : 4 : : 6 : 12 we have 2+4 : 6+12 : : 2 : 6 and 4—2 : 12—6 : : 4 : 12. Apply the test and both will be found true.

69. *In every continued proportion, the sum of any number of antecedents is to the sum of the same number of consequents, as one antecedent is to its consequent.* Thus if we have the continued proportion 2 : 4 : : 6 : 12 : : 8 : 16 then we say 2+6+8 : 4+12+16 : : 2 : 4. Apply the test and this will be found true. The same might be proved of any number of equal ratios.

OF PROPORTIONAL LINES.

70. Theorem.—*If a line be drawn through two sides of a triangle parallel to the third side, it divides those two sides proportionally.* More depends upon this proposition than upon any other in geometry, and as the demonstration is long, we shall divide it into three parts. What we are to prove is, that if the straight line D E (fig. 37) be drawn through the two sides A B, F 37
A C, of the triangle A B C parallel to the third side or *base* B C, then the two sides A B, A C are divided proportionally at the points D and E ; and we shall have the proportion

A B : A D : : A C : A E.

Dem. 1.—*First* we say that if A P (fig. 36) be divided F 36
into equal parts, and if lines be drawn through the points of division parallel to the base P R, then A R will be divided into the same number of equal parts as A P ; in other words A F=F G=G H=H I, &c. Through the points F, G, H, &c. draw the lines F K, G L, H M, &c. parallel to A P. Then the triangles A B F, F K G, G L H, &c. are equal. Why ? Because they all have a side and two adjacent angles equal (55): namely, A B=F K=G L, &c. because parallels comprehended between parallels are equal (38); the angle B A F=K F G=L G H, &c. because internal-external angles are equal (34); and lastly A B F=F K G=G L H, &c. because angles

which have their sides parallel and directed the same way are equal (39). Therefore A F=F G=G H, &c. and A R is divided into the same number of equal parts as A P. DEM. 2.—*Secondly* we say that if the
F 36 whole side A P (fig. 36) is to a part A E in the ratio of two whole numbers, and if through the point E the line E I be drawn parallel to the base P R, then the other side A R will be to its part A I in the same ratio; and we shall have the proportion.

A P : A E : : A R : A I.

If A P is divided, for example, into 7 equal parts, and A E contains 4 of them, then we shall have the proportion

A P : A E : : 7 : 4.

But, by the first part of the demonstration A R will also be divided into 7 equal parts, of which A I contains 4. Then we have the proportion

A R : A I : : 7 : 4.

Whence, leaving out of the two proportions the common ratio 7 : 4 (64) we have the required proportion

A P : A E : : A R : A I.

The same would be true for any other numbers. DEM. 3.—*Thirdly* we say that if the whole side A B
F 37 (fig. 37) is not to a part A D in the ratio of two whole numbers, still, if D E be drawn parallel to the base B C, the two sides A B, A C will be divided proportionally, and we shall have universally the proportion

A B : A D : : A C : A E.

The method of proof is by what is called a *reductio ad absurdum*, and is as follows. If the fourth term of the above proportion be not A E, it must be some line either greater or less than A E. Now if we can show that it is absurd to suppose it either greater or less than A E, the other terms remaining the same, then the fourth term must be A E, and the proportion will be true. Let us then in the first place take a fourth term less than A E, for instance, A O. Then the proportion will be

A B : A D : : A C : A O.

Now suppose A C divided as in the first part of the *demonstration* into such a number of equal parts that each shall be less than O E. Then one point of division will be between E and O. Let that point be G, and draw G F parallel to B C. Then by the second

part of the demonstration we shall have the proportion

A B : A F : : A C : A G.

But we had A B : A D : : A C : A O.

Inverting the means in both (65) we shall have

A B : A C : : A F : A G.

A B : A C : : A D : A O.

Whence leaving out the common ratio A B : A C (64) we have A F : A G : : A D : A O.

This proportion contains an absurdity as will be seen by inverting the means; for then

A F : A D : : A G : A O,

that is, a less is to a greater as a greater is to a less, which is manifestly absurd. This absurdity arises from supposing that the fourth term could be less than A E. Therefore we know that it cannot be less. If now we should suppose it greater, and take O on the other side of E, by repeating the same course of reasoning verbatim, we should arrive at a similar absurdity. The fourth term then cannot be greater than A E. And since it can be neither less nor greater than A E, it must be A E, and the proportion above enunciated is rigorously true, namely

A B : A D : : A C : A E.

Schol.—From the proportion A B : A D : : A C : A E we obtain several others which will be used hereafter. Since B D is the difference of the two first terms, and C E the difference of the two last, we have (68) these two

B D : C E : : A B : A C

B D : C E : : A D : A E.

Inverting the means (65) we have these two

B D : A B : : C E : A C

B D : A D : : C E : A E.

71. Theorem.—*If a line be drawn so as to divide two sides of a triangle proportionally, that line is parallel to the third side.* Thus if D E (fig. 38) be so drawn F 38
that we have the proportion

B D : D A : : C E : E A,

then we say that D E is parallel to B C. Dem.—The method is by *reductio ad absurdum.* If D E is not parallel to B C, some other line drawn through D must be. Suppose that line to be D F. Then if D F is

parallel to the base B C, we have by the preceding

B D : D A : : C F : F A.

But by hypothesis B D : D A : : C E : E A, whence leaving out the common ratio (64) we have

C F : F A : : C E : E A.

This proportion is absurd as will be seen by inverting the means. Then we have

C F : C E : : F A : E A.

That is, a greater is to a less as a less is to a greater, which is manifestly absurd. Therefore no line different from D E, can be drawn through D parallel to B C.

72. Problem.—*To find a fourth proportional to three given lines.* Sol.—Let the three lines be A, B,
F 39 C (fig. 39). Draw the indefinite lines D P, D R, making any angle at pleasure. Take D E=A, D F =B, and D G=C. Join E and F, and through G draw G H parallel to E F. Then D H will be the fourth term required. For since G H is parallel to the base E F (70) we have the proportion

D E : D F : : D G : D H.

This geometrical operation corresponds to the *Single Rule of Three* in arithmetic.

73. Problem.—*To divide a given straight line into any number of equal parts.* Suppose it were re-
F 40 quired to divide A B (fig. 40) into six equal parts. Sol.—Draw the line A P indefinitely. Take A C of any convenient length, and apply it six times to A P. Through H, the last point of division, draw H B. Through C draw C I parallel to H B. A I will be a sixth part of A B. For since C I is parallel to the base H B, we have (70) A I : A B : : A C : A H. But by construction A C is one sixth of A H. Therefore A I is one sixth of A B. Apply A I six times to A B, and A B will be divided into six equal parts.

74. Problem.—*To divide a given line into parts proportional to any given lines.* Suppose it were re-
F 41 quired to divide D F (fig. 41) into three parts proportional to the three given lines A, B, C. Sol.—Draw D P indefinitely. Take D G=A, G H=B, H L=C. Draw L F, and through H and G draw H K and G I parallel to L F. Then calling H K the base, since G I is parallel to it we have (70)

D I : I K : : D G : G H.

But D G=A and G H=B. Therefore

D I : I K : : A : B.

Again calling H K the base, we have (64)

D K : D H : : I K : G H.

But calling L F the base, since H K is parallel to it, we have D K : D H : : K F : H L, whence leaving out the common ratio

I K : G H : : K F : H L

or (65) I K : K F : : G H : H L.

But G H=B and H L=C. Therefore

I K : K F : : B : C.

And the line D F is divided as required.

75. PROBLEM.—*To divide one side of a triangle into two parts proportional to the other two sides.* Suppose it were required to divide B C (fig. 42) into two parts proportional to A B and A C. SOL.—Draw A D so as to bisect the angle B A C, and D will be the point of division. For we shall have the proportion CD : DB : : CA : AB. F 42

Why? Draw B E parallel to A D till it meets C A produced. Then since A D is parallel to the base E B (70) we have

C D : D B : : C A : A E.

But A E=A B. For the angle B E A=D A C (34) and E B A=B A D. Now B A D=D A C by construction. Therefore A E B=A B E. Then (50) A E=A B. Substituting A B for A E in the last proportion, we have C D : D B : : C A : A B.

76. PROBLEM.—*Through a given point in an angle, to draw a line so that the parts intercepted between the point and the sides of the angle shall be equal.* Suppose it were required to draw through the point B in the angle D A E (fig. 43) a line D E in such a manner that D B should be equal to B E. SOL.—Draw B C parallel to A E. Take C D=C A, and through the points D, B, draw D E. Then D B=B E. For since B C is parallel to the base A E (70) we have F 43

D B : B E : : D C : C A.

But D C was taken equal to C A. Therefore D B=B E.

SIMILAR TRIANGLES.

77. DEF.—*Two triangles are said to be similar when they are equiangular with respect to each other.*

F 44 Thus if A B C (fig. 44.)=C D E, B A C=D C E, and A C B=C E D, then the triangles A B C and C D E are similar. Now there are three cases in which two triangles are equiangular. 1.—*When they have their sides parallel each to each.* For if the triangle B A C
F 35 (fig. 35) has its sides respectively parallel to the sides of the triangle E D F, then the angle B=the angle E; the angle A=the angle D; and the angle C=the angle
F 39 F (fig. 39). 2.—*When they have their sides perpendicular each to each.* For then by turning one of the triangles about one of its vertices by the space of a quadrant, the sides will become parallel each to each. 3.—*When they have an angle of the one equal to an angle of the other, and the sides including these angles proportional.* Thus if the angle A=A (fig. 45), and if

A B : A D :: A C : A F,

then we say the triangles are equiangular. For if the side A D be placed upon A B, since the angles at A are equal, A F will fall upon A C. Then, from the proportion A B : A D :: A C : A F
D F must be parallel to B C (71). Consequently the angle A D F=A B C and A F D=A C B (34), and the two triangles are equiangular. In the above three cases, then, according to the definition, two triangles are similar.

78. Theorem.—*Two similar triangles have their homologous sides proportional.* By *homologous* sides, we mean those which have corresponding positions with respect to the equal angles. Thus in the similar
F 44 triangles B A C and D C E (fig. 44) A B is homologous to C D, being opposite to equal angles, and so of the rest. We are now to demonstrate the following proportion

A B : C D :: B C : D E :: A C : C E.

Dem.—Let the two triangles be so placed that A E and C E shall be in the same straight line. Produce A B and E D till they meet in F. Now B C is parallel to E F, because the angle B C A=D E C (34). Also C D is parallel to A F, because the angle D C E= B A C. Then since B F is parallel to the base E F we have (70)

A C : C E :: A B : B F.

But B F=C D (38). Therefore

A C : C E :: A B : C D.

Again, since C D is parallel to the base A F

A C : C E :: F D : D E.

But F D=B C. Therefore

A C : C E :: B C : D E.

Thus the three ratios formed by the three couples of homologous sides are equal, and give the continued proportion

A B : C D :: B C : D E :: A C : C E.

79. THEOREM.—*If two triangles have their homologous sides proportional, they are similar.* If the triangles A B C and D E F (fig. 46) give the proportion F 46

A B : D E :: B C : E F :: A C : D F,

then we say they are similar. This proposition is the converse of the preceding. DEM.—Draw D G so as to make the angle G D F=A. Draw F G so as to make the angle D F G=C. Then the angle G=B (48), and the triangles A B C and D G F are similar by construction. Now if we prove that the triangle E D F is equal to D G F, it will follow that E D F is similar to A B C. By the conditions we have

A C :: D F :: A B : D E,

and by construction we have (78)

A C : D F :: A B : D G.

In these proportions the three first terms are the same, and therefore (63) the fourth terms must be equal. Thus D E=D G. Again, by the conditions we have

A C : D F :: B C : E F,

and by construction,

A C : D F :: B C : F G.

Therefore E F=F G. Then the triangles E D F and D F G are equal (57) having their three sides respectively equal. Consequently E D F is similar to A B C, which was to be demonstrated.

80. THEOREM.—*If from any point in a semicircumference a line be drawn perpendicular to the diameter, it will be a mean proportional between the two segments of the diameter.* Thus we say that A D (fig. 47) is a mean proportional between B D and D C, F 47 or in other words that

B D : A D :: A D : D C.

DEM.—Draw the two chords A B and A C. Then the two triangles A B D and A D C will be similar. For the angle A D B=A D C, being right angles. Also since B A C is a right angle (42) being inscribed

in a semicircumference, B A D is a complement of D A C. But B A D is likewise a complement of D B A, since (47) the acute angles in a right triangle are complements of each other. Therefore (21) D A C= D B A, having the same complement D A B; and as the third angles must be equal (48), the two triangles are similar. We have then B D homologous to A D, and A D homologous to D C. Therefore

B D : A D : : A D : D C.

Schol.—The problem *to find a mean proportional between two lines* is thus solved. Let the two lines be B D and D C. Join these two for a diameter as B C. Describe the semicircumference, and at the point of junction D, erect the perpendicular D A. This will be the mean proportional sought.

81. Theorem.—*If from a point without a circle a tangent and secant be drawn, the tangent will be a mean proportional between the entire secant and the*
F 48 *part without the circle.* Thus if O (fig. 48) be the point, O A the tangent, and O C the secant (40), then we say that O A is a mean proportional between O D and O C, that is,

O D : O A : : O A : O C.

Dem.—Draw the chord A D. Then the triangles O A D and O A C will be similar. For the angle O is common to both, and O A D=A C D, since both have for their measure half the arc A D (41). Therefore the remaining angles are equal (48), and the triangles are similar. Then O D is homologous to O A, and O A is homologous to O C, and we have

O D : O A : : O A : O C.

82. Problem.—*To divide a given line in extreme and mean ratio.* By this we mean to divide a line into two such parts that the greater part shall be a mean proportional between the whole and the less.
F 49 Thus A B (fig. 49) will be divided in extreme and mean ratio, if we can find a point F such that

B F : A F : : A F : A B.

The question then is to find the point F. Sol.—Erect the perpendicular D C=half of A B, and with C as a centre, and C B as radius describe a circle. Through A and C draw the secant A E. Then with A as a centre and A D as a radius describe an arc cutting A B in

F. F will be the point sought. For by the preceding proposition we have

A D : A B : : A B : A E.

Then (68)

A B—A D : A E—A B : : A D : A B.

But A B—A D=B F, and A E—A B=A E—D E=A D=A F. Therefore

B F : A F : : A F : A B,

which is the proportion sought.

QUADRILATERALS.

83. Definitions.—*A Quadrilateral is any figure bounded by four straight lines.*

A Parallelogram is a quadrilateral, which has its opposite sides equal and parallel. Thus A B C D (figs. 53, 54, 55, 56).

A Square is a right parallelogram, having all its sides equal, and all its angles right angles. Thus A B C D (fig. 53). F 53

An Oblong is a right parallelogram, having its opposite sides equal, and all its angles right angles. Thus A B C D (fig. 54). F 54

A Rhombus is an oblique parallelogram, having all its sides equal, and none of its angles right angles. Thus A B C D (fig. 55). F 55

A Rhomboid is an oblique parallelogram, having its opposite sides equal and none of its angles right angles. Thus A B C D (fig. 56). F 56

A Trapezoid is a quadrilateral, which has only two of its opposite sides parallel. Thus A B C D (fig. 51). F 51

A Trapezium is a quadrilateral, which has no two of its opposite sides parallel. Thus A B C D (fig. 52). F 52

The Diagonal of a parallelogram is a straight line drawn through two opposite vertices. Thus B D (fig. 56). F 56

84. Cor. I.—*The Diagonal of a parallelogram divides it into two equal triangles.* Thus the triangles A B D and B C D (fig. 56) are equal. For by definition A B=D C, and A D=B C, and B D is common to both. Therefore (57) the triangles are equal. F 56

Cor. II.—*The opposite angles of a parallelogram are equal.* Thus the angle A B C=A D F (fig. 55)

because the triangle A B C=the triangle A D C. Also the angle B A D=B C D, because the triangle B A D= the triangle B C D. Cor. III.—*The two diagonals of a parallelogram mutually bisect each other.* Thus
F 55 B D and A C are bisected in E (fig. 55). For A B= D C by definition. The angle A B E=E D C being alternate-internal. And the angle B A E=A C D for the same reason. Therefore the triangles A B E and E D C are equal (55), and all their parts are equal. Hence A E=E C and B E=E D.

POLYGONS.

85. Def.—*Polygon, is the general name for every figure bounded by straight lines.* Accordingly we might have treated of triangles and quadrilaterals under this head. But we thought it more useful as well as more simple, to consider them separately. Polygons are divided into *regular* and *irregular. Regular polygons are those which have all their sides equal, and all their angles equal.* Thus an equilateral triangle and a square are regular polygons. *Irregular polygons are such as do not possess both these properties.—Similar polygons are those which have their angles equal, each to each, and their homologous sides proportional.* The student must observe that two polygons may be equiangular *with respect to each other*, when neither is equiangular *considered by itself.* Thus a regular polygon is equiangular in itself, but two similar polygons are not necessarily so, though they are equiangular with respect to each other. There are particular names for polygons depending on the number of sides. Thus a *pentagon* is a polygon of five sides; a *hexagon* one of six sides; a *decagon* one of ten sides, &c. But we shall use the general term polygon, unless where the necessity of the case requires us to be more specific.

86. Theorem.—*The sum of the interior angles of any polygon is equal to as many times two right angles as there are sides minus two.* Dem.—From the vertex A
F 57 (fig. 57) draw the diagonals A C, A D, A E to all the vertices not adjacent. Then the polygon will be divided into as many triangles as there are sides minus two.

Thus if the polygon have six sides, there will be four triangles, if seven sides, five, and so on. Now the sum of the angles of each of the triangles is equal to two right angles (46). Therefore there will be as many times two right angles in the polygon as there are triangles minus two; that is, as there are sides minus two.

SCHOL.—Let it be observed that we here speak only of *convex* polygons, that is, of those whose vertices are all directed *outward* as in the figure.

87. THEOREM.—*If two polygons are composed of the same number of similar triangles similarly disposed, the polygons are similar.* Thus if the two polygons (fig. 58) are composed of the same number of similar triangles, then we say they have their angles equal, each to each, and their homologous sides proportional. F 58

DEM.—Because A B C is similar to F G H; the angle B=G, and (78)

A B : F G : : B C : G H.

Again the angle B C D=G H I, because from the similarity of the successive triangles, B C A=G H F and A C D=F H I; while from the two proportions

B C : G H : : A C : F H

and A C : F H : : C D : H I,

we have (64) B C : G H : : C D : H I.

Thus far then, we have the angles equal, each to each, and the homologous sides proportional, and it is evident that the same reasoning might be continued as long as there were similar triangles placed in the same order.

88. PROBLEM.—*Upon a given line to construct a polygon similar to a given polygon.* SOL.—Suppose F G (fig. 58) were the given line and A B C D E the given polygon. Consider F G homologous, for example, to A B. Then draw G H making the angle F G H=A B C, and draw F H making the angle G F H=B A C. The triangles A B C and F G H will be similar. Again draw H I making the angle F H I=A C D, and draw F I making the angle H F I=C A D. The triangles A C D and F H I will be similar. Proceed in this manner till the construction is completed, and the two polygons will be similar by the preceding proposition. F 58

89. THEOREM.—*Two regular polygons of the same number of sides are similar.* DEM.—Suppose the two

F 57 polygons are regular hexagons (fig. 57). Then we say, in the first place, they are equiangular with respect to each other, for each of the angles in both polygons is equal to one sixth of eight right angles (86). Again their homologous sides are proportional. For, by the definition of regular polygons (85) A B=B C=C D, &c. and G H=H I=I K, &c. Therefore, whatever be the ratio of A B to G H, the same must be the ratio of B C to H I, of C D to I K, &c., that is,

A B : G H : : B C : H I : : C D : I K,

and so on round the figure. Hence the two polygons are similar. The same reasoning would apply to any other number of sides.

90. Theorem.—*Every regular polygon may be inscribed in a circle, and circumscribed about a circle.* A polygon is said to be *inscribed*, when all its vertices are in the circumference, and to be *circumscribed*, when all its sides are tangents. Dem. 1.—Let there be a
F 59 regular polygon A B C D E F (fig. 59). Find the centre I of a circle (29) to pass through the three points B, C, D. We say the same will pass through all the other vertices. First it will pass through E. Draw the chords B D and C E. Then, by the definition, the triangles B C D and C D E are equal (53), and if C D were placed upon B C, D E would fall upon C D. Accordingly the same circle which passes through B, C, D, will also pass through C, D, E. The same reasoning will apply to F and A, and to any number of vertices. 2.—Secondly, we say that this polygon may be circumscribed. Draw I H perpendicular to the middle of A B (28) and I G perpendicular to the middle of A F. Describe a circle with the radius I H, and A B will be a tangent (40). Now we say that A F will be a tangent to the same circle. Because the two right angles A I H and A I G are equal (59) since they have the hypothenuse A I common, and A H half of A B=A G half of A F. Therefore I H=I G, and A F is a tangent. The same might be proved in like manner of all the other sides. Thus whenever a regular polygon is given, there may be a circle circumscribed about it, and a circle inscribed in it, or, in the words of the enunciation, the polygon may be inscribed and circumscribed.

91. Problem.—*To inscribe a square in a given circle.* We cannot solve the general problem, having a given circle to inscribe in it a regular polygon of any number of sides, since we have no means of dividing the circumference of a circle into any given number of equal parts. But there are certain particular cases in which the solution is possible. We begin with the square. Sol.—Let the given circle be A B C D (fig. 60). Draw two diameters perpendicular to each other, and join their extremities by chords. A B C D is a square (83), because its sides are equal (53) and its angles are right angles (42). F 60

92. Problem.—*To inscribe in a given circle a regular hexagon and an equilateral triangle.* Sol.—Take the radius A O (fig. 61) in the compasses, and apply it round the circumference. We say that it will be contained exactly six times, or, in other words, that *the side of an inscribed hexagon is equal to radius.* Since A O=B O, the angle O A B=O B A. Then, supposing A B to be the side of a regular hexagon, the angle A O B must be equal to 60°, since the arc A B is a sixth part of the whole circumference. Then the angles O A B+O B A must be equal to 120° (46), and since they are equal, each must be 60°. Therefore the triangle A O B is equilateral (51), and A B, the side of a hexagon, is equal to the radius A O. If now we would inscribe an equilateral triangle, it is only necessary to join the alternate vertices, A, C, E. Indeed we may remark generally that when any polygon of an even number of sides is already inscribed, we may always inscribe one of half the number of sides, by joining the alternate vertices. Also, by bisecting the arcs, whether an even number or not, and drawing chords to the half arcs, we may always inscribe one of double the number of sides. F 61

93. Problem.—*To inscribe in a given circle a regular polygon of ten and one of fifteen sides.* Sol.—First, to inscribe one of ten sides. Divide the radius O A (fig. 62) in extreme and mean ratio (82). Let O M be the greater part. Take the chord A B=O M, and apply it round the circle. We say it will be contained exactly ten times, or in other words that the arc A B is a tenth part of the circumference. To F 62

prove this we need only show that the angle A O B= 36°. We have by construction,

A M : M O : : M O : A O,

or, drawing B M and substituting A B for M O,

A M : A B : : A B : A O.

Then the triangles A M B and A O B are similar (77) having the angle A common, and the sides including it proportional. But A O B is isosceles. Therefore A M B is also isosceles, and B M=A B=O M. This makes O M B isosceles, and the angle M O B=M B O. Now B M A, being an exterior angle (49) is equal to the two opposite interior angles M O B+M B O= twice A O B. Then B A M=B M A=twice A O B, and O B A=M A B=twice A O B. Hence all the angles of the triangle A O B or 180°=five times A O B. Then A O B=one fifth of 180°=36°, and A B is the side of a regular decagon. 2.—If now, *secondly*, we wish to inscribe a regular polygon of 15 sides, we have only to find one fifteenth of a circumference. For this purpose, let A L be the side of a hexagon and A B that of a decagon. Then B L will be the arc required, for B L=$\frac{1}{6}-\frac{1}{10}$ of a circumference, that is, one fifteenth. Lastly, by joining the alternate vertices of a decagon we should have a pentagon; and by bisecting the arcs which are one fifteenth and drawing chords, we should have a polygon of 30 sides, and so on indefinitely.

94. Theorem.—*The circle is a regular polygon of an infinite number of sides.* Dem.—Inscribe in the
F 63 circle (fig. 63) any one of the regular polygons before mentioned, for instance, a hexagon, as A B C D E F. Bisect the arcs B C, C D, &c., and join the half arcs by the chords B H, H C, C I, &c. Thus you have a regular polygon of 12 sides. Proceed in the same manner with this, and you have one of 24 sides, then one of 48 sides, and so on without limit. Now it is obvious that the polygon of 12 sides approaches nearer to a coincidence with the circle, than that of six sides. In the same manner the polygon of 24 sides approaches nearer than that of 12, and the polygon of 48 sides approaches nearer than that of 24, and so on without a limit. But the difference between the first polygon and the circle is a finite or limited quantity, and we have seen that this difference constantly diminishes as we

increase the sides. Accordingly if the number of sides were increased to infinity, the difference would become nothing; for no one can doubt that the endless diminution of a limited quantity must bring it to nothing. Thus the polygon of an infinite number of sides would not differ from a circle. This idea of a circle agrees with the definition before given of a curved line (10) namely, that it is made up of infinitely small straight lines.

95. Theorem.—*The perimeters of regular polygons of the same number of sides are to each other as the radii of their circumscribed circles.* By the *perimeter* of a polygon we mean the sum of its sides. Then we say that the perimeter A B C D E F (fig. 64), is to F 64
the perimeter G H I K L M as C N is to I O. Dem.—Suppose the two polygons are hexagons. As they are similar (89) we have

B C : H I : : C D : I K.

Then (66)

6 times B C : 6 times H I : : C D : I K.

But 6 times B C is the perimeter of the first polygon, and 6 times H I is the perimeter of the second. Moreover the triangles B N C and H O I are similar. For the angle B N C=H O I since the arcs B C and H I contain the same number of degrees, and B C N=H I O (42) being inscribed in segments containing the same number of degrees. Therefore

C D : I K : : C N : I O.

Accordingly by making the substitutions in the proportion, 6 times B C : 6 times H I : : C D : I K, we have the following;

perim. A B C D E F : perim. G H I K L M : : C N : I O.

As the same reasoning might be employed for any other number of sides than 6, the proposition is demonstrated.

96. Theorem.—*The circumferences of circles are to each other as their radii.* Dem.—This follows directly from the two last propositions, for the circumferences of circles are the perimeters of regular polygons of an infinite, and therefore *the same* number of sides. Moreover the radii of the circumscribed circles become, in this case, the radii of the circles to be compared, the polygons being confounded with the circumscribed circles. Cor.—*Similar arcs are to each other as their radii.* By *similar arcs* we understand those

which contain the same number of degrees or measure equal angles at the centre. Now from the definition of a degree (15) such arcs are to each other as the circumferences of which they are a part. But these last are to each other as their radii. Therefore similar arcs are to each other as their radii.

SECTION SECOND.

SURFACES.

97. DEF.—*By the word Surface we understand, in the abstract, that magnitude which has length and breadth without thickness.* But a more definite idea will be obtained if we introduce motion. Accordingly we may say—*a surface is the space described by a line moving any other way than lengthwise.* Thus we have the origin of the two dimensions. For the line itself has one dimension, namely, *length*, and its motion makes another, namely, *breadth.* Speaklng abstractly there is no thickness. But as you cannot make obvious to the senses, that which has absolutely no thickness, it is sufficiently near the truth to say—*a surface has length and breadth with an infinitely small thickness.* This is analogous to our definition of a line (2), for the infinitely small breadth and thickness of the moving line, would give an infinitely small thickness to the generated surface. Moreover as the boundaries of a line were points, so now, for a similar reason, the boundaries of a surface are lines.

98. DEF.—*A plane surface is that with which a straight line will coincide in every direction.* Thus, if we leave its thickness out of consideration, a sheet of paper perfectly smooth and even, may be taken to represent a plane surface, for in whatever direction we apply the straight edge of a rule to it, the rule will touch it in every point. Such a surface is usually designated by the word *plane* alone.—*A polygonal surface is one which is composed of several planes.* If a surface is neither plane nor composed of planes, it is a curved surface. But in order to give a definition which

may make a plane the element of all surfaces, we say—*a curved surface is one which is composed of infinitely small planes*. This is the point of view in which we shall consider it hereafter.

99. SCHOL.—*Quantity of Surface. How estimated. Unit of Surface. Area.* In the first section we considered figures only with reference to the lines and angles of which they consist, and we called those figures *equal*, which, being applied the one to the other coincide throughout. In the present section we are to consider figures with reference to the *quantity of surface*, which they embrace, and we shall call those figures *equivalent*, which embrace equal quantities of surface. The question then arises, how is this quantity of surface to be estimated? In other words, how are surfaces measured and compared? In measuring and comparing lines, (7) we found it necessary to fix upon some quantity of the same kind as a standard of measure, and we called this a *linear* unit. In like manner if we would measure and compare surfaces, we must fix upon some quantity of the same kind, to serve as a *unit of surface* or *superficial unit*. DEF.—*A superficial unit is a square whose side is a linear unit.* For in order to be of the same kind, this unit must have two dimensions, length and breadth; and as that is obviously most simple, in which these two dimensions are the same, geometers have unversally, adopted a square, which is the most regular of figures, having its sides all equal and its angles all equal. Accordingly we express the measure of surfaces, by stating the number of square inches, square feet, square yards, &c., which they contain; meaning thereby the number of squares whose side is an inch, a foot, a yard, &c. The measure of a figure thus expressed, is usually called its *area*.

100. THEOREM.—*The area of a right parallelogram is equal to the product of its base by its altitude.* By the *altitude* of a parallelogram we mean a perpendicular let fall from one side to another parallel side. By the *base* we mean the side upon which the perpendicular falls. Thus E F (fig. 50) is the altitude, and F 50
A D the base. In case of a right parallelogram as A B C D (fig. 65) it follows from the above definition F 65
that A B is the altitude and A D the base. Then we say that the area of A B C D$=$A B$\times$A D. That is

A B C D contains as many superficial units as the product of the linear units in A D by those in A B. Dem.—The superficial unit, as we have just seen, will depend upon the linear unit. Suppose then A D=8 inches and A B=6 inches. Here the superficial unit will be a square inch, as E F G H, and we are to shew that it is contained in A B C D, 6 times 8 or 48 times. If we mark the inches in A D and erect perpendiculars, and do the same with A B, as in the figure, each row will contain as many squares as there are inches in A D, that is 8; and there will be as many of these rows as there are inches in A B, that is 6. The whole number then is 6 times 8 or 48. The proposition is equally true if A D and A B do not contain an exact number of inches. If for example A D$=6\frac{1}{2}$ inches and A B$=4\frac{1}{3}$ inches, still we say that A B C D$=6\frac{1}{2}\times4\frac{1}{3}=28\frac{1}{6}$ square inches. For since A D$=6\frac{1}{2}=\frac{13}{2}=\frac{39}{6}$; and A B$=4\frac{1}{3}=\frac{13}{3}=\frac{26}{6}$, we may suppose A D divided into 39 parts each equal to $\frac{1}{6}$ of an inch, and A B divided into 26 equal parts of the same value. Then by erecting perpendiculars as before, we shall have A B C D$=39\times26=1014$ squares, the side of which is $\frac{1}{6}$ of an inch. Now of these small squares a square inch contains $6\times6=36$, since each row contains 6 squares and there are 6 rows. Accordingly if we divide the whole number 1014 by 36 the number in a square inch, we shall have the number of square inches. Now $\frac{1014}{36}=28\frac{1}{6}$ which was to be proved. Accordingly, since the same reasoning might be employed for any other values of A D and A B, we conclude universally that the area of a right parallelogram is equal to the product of its base by its altitude. Cor.—*The area of a square is found by multiplying one of its sides by itself.* For by the definition, the base and altitude are the same. And hence the origin of the term *square* as used in arithmetic and algebra, to designate the product of a number multiplied by itself.

101. Theorem.—*The area of any parallelogram is equal to the product of its base by its altitude.* If a parallelogram is not right, it is oblique. Now if we prove that an oblique parallelogram is *equivalent* to a right parallelogram of the same base and altitude, it will follow that it must have the same measure, namely, its base into its altitude. Accordingly let A B E F

(fig. 66) be a right parallelogram, and A B C D an oblique one, of the same base A B and the same altitude B E. We say they are equivalent. Dem.—The right triangle A F D=B E C (59), since A F=B E and A D=B C from the nature of parallelograms, and the two triangles have the hypothenuse and a side of the one, equal to the hypothenuse and a side of the other. Now if from the whole figure A B C F, we take A F D, there will remain A B C D. Again if from the whole figure we take B E C, there will remain A B E F. But it is an axiom that if equals be taken from the same thing, equivalents will remain. Therefore A B C D= A B E F, and the area of A B C D=A B$\times$B E, this being the measure of A B E F (100). F 66

102. Theorem.—*The area of any triangle is equal to half the product of its base by its altitude.* By the *altitude* of a triangle we mean a perpendicular let fall from one of the vertices to the opposite side, produced if necessary; and by the *base* the side upon which the perpendicular falls. Thus in the triangle A C D (fig. 67) C E is the altitude and A D the base. Then we say that the area of A C D=half of A D$\times$C E. Dem.—The triangle A C D=half the parallelogram A B C D of the same base and altitude (84). But the area of A B C D=A D$\times$C E (101). Therefore the area of A C D=half of A D$\times$C E. F 67

103. Theorem.—*The area of a trapezoid is equal to the product of its altitude by half the sum of its parallel sides.* By the altitude of a trapezoid we mean the perpendicular let fall from one of the parallel sides to the other. Thus in the trapezoid A B C D (fig. 68), C E is the altitude, B C and A D being the parallel sides. Then we say that the area of A B C D=C E$\times$ half of (A D+B C). Dem.—The diagonal A C divides the trapezoid into two triangles having the same altitude as the trapezoid, namely C E. Now the area of A C D=C E$\times$half of A D (102). Also the area of A B C, taking A for the vertex and B C for the base, is equal to C E$\times$half of B C, since C E=A F (38). Therefore, since the trapezoid is equal to the sum of the triangles, its area must be C E$\times$half of A D+C E$\times$ half of B C, or C E$\times$half of (A B+B C).

104. Theorem.—*The area of a regular polygon is equal to the product of its perimeter by half the radius*

F 64 *of the inscribed circle.* Let A B C D E F (fig. 64) be the polygon, and N P the radius of the inscribed circle. Then we say that the area is equal to (B C+C D+D D+E F+F A+A B)$\times\frac{1}{2}$ N P. Dem.—From the centre N draw lines N B, N C, &c. to all the vertices. These will all be equal, being radii of the circumscribed circle. Then the polygon will be divided into as many equal triangles as it has sides. Moreover these triangles have for their common altitude the radius N P of the inscribed circle, and the sum of their bases B C+C D+D E,+&c. is the perimeter of the polygon. Therefore, adding their measures (102), we have for the area of the polygon its perimeter multiplied by half the radius of the inscribed circle.

105. Theorem.—*The area of a circle is equal to the product of its circumference by half the radius.* Dem.—This follows directly from the preceding. For the circumference of the circle is the perimeter of a regular polygon of an infinite number of sides, and the radius of the inscribed circle is its own radius.

106. Theorem.—*The area of a sector is equal to its arc multiplied by half the radius.* Let C A M B
F 69 (fig. 69) be the sector. Then we say its area is equal to the arc A M B$\times\frac{1}{2}$ A C. Dem.—Suppose the arc A M B to be made up of infinitely small straight lines, and radii drawn from C to each of the points. Then the sector would be divided into triangles, the sum of whose bases would be the arc A M B and whose common altitude would be the radius A C. Therefore the area of the sector, being the sum of the areas of these triangles, is A M B$\times$half of C A. Scho.—*The area of a segment is found by taking the difference between the area of a triangle and that of a sector.* Thus if we would find the area of the segment formed by the arc A M B and the chord A B, we have only to subtract the area of the triangle C A B from the area of the sector C A M B.

107. Theorem.—*Every polygon may be converted*
F 70 *into an equivalent triangle.* Let A B C D E (fig. 70) be the polygon. Dem.—Draw the diagonal C E, and through D draw D F parallel to C E to meet A E produced. Then draw C F, and the triangle C E F will be equivalent to C E D. Why? Because they have the same base C E, and their altitudes are equal, since

their vertices F and D are in a line parallel to C E. Consequently, having the same measure, the triangles are equivalent. Then by leaving out C D E and taking C E F in its stead, we have the quadrilateral A B C F equivalent to the pentagon A B C D E. In the same manner we may leave out the triangle A B C and take an equivalent one A G C in its stead. Then the triangle G C F=the quadrilateral A B C F=the pentagon A B C D E. The same process would apply to any number of sides. SCHO.—*How to measure an irregular polygon.* This may be done in two ways. *First,* by drawing diagonals as in A B C D E (fig. 85), F 85
the polygon is divided into triangles, which may be measured separately; and the sum of their areas will be the area of the polygon. *Secondly.* the polygon may be converted into an equivalent triangle, by the method just explained; and the area of this triangle will be the area of the polygon.

108. THEOREM.—*The square described upon the hypothenuse of a right triangle is equivalent to the sum of the squares described upon the other two sides.* This is the celebrated proposition, with the discovery of which Pythagoras is said to have been so delighted, that he sacrificed a hundred oxen to the Muses. We are to prove that the square B C G F (fig. 71) is equiva- F 71
lent to the sum of the squares A B H L and A C I K, or more briefly that $BC^2 = BA^2 + AC^2$. DEM.—From A, the vertex of the right angle, let fall the perpendicular A D and produce it to E. The square B G is thus divided into two right parallelograms B E and C E. If then we prove that B E=the square A H, and C E= the square A I the proposition will be demonstrated. Draw the diagonals A F and H C. Thus we have two triangles A B F and H B C. These are equal. Why? Because the angle A B F=H B C, since each is equal to a right angle plus the angle A B C. Also A B= B H and B F=B C, from the definition of a square. Therefore (53) the triangle A B F=H B C. But H B C=half the square A H since it has half its measure (102), namely H B$\times$half of A B. Again A B F=half of B E, since it has half its measure, namely B F$\times$half of B D. Now if two halves are equivalent, their wholes must be equivalent; that is, the square A H=the oblong B E. In precisely the same

manner we might prove that the square A I=the oblong C E. But the two oblongs B E and C E make the square B G. Therefore B G=A H+A I or $BC^2 = AB^2 + AC^2$, which was to be demonstrated. SCHO.—*When two sides of a right triangle are known, the third can always be found by extracting the square root of the sum or difference of the squares of the other two sides.* From the above equation $BC^2 = AB^2 + AC^2$ we have $AB^2 = BC^2 - AC^2$ and $AC^2 = BC^2 - AB^2$. Also by extracting the square root, we have $BC = (AB^2 + AC^2)^{\frac{1}{2}}$, $AB = (BC^2 - AC^2)^{\frac{1}{2}}$, and $AC = (BC^2 - AB^2)^{\frac{1}{2}}$.

109. PROBLEM.—*To make a square equal to the sum or the difference of two given squares.* Suppose A (fig. 72) is the side of one of the given squares and B that of the other. SOL.—*First*, to make a square equal to their sum, take E D=A, at E erect a perpendicular E F, and take E G=B. Then G D will be the side of the square required. For by the preceding proposition $GD^2 = GE^2 + ED^2$. *Secondly*, to make a square equal to the difference of the squares upon A and B, Supposing A the greater, make a right angle G E H and take E G=B. Then with G as a centre and a radius equal to A, describe an arc cutting the other side in a point H. E H will be the side of the square required. For (108) $EH^2 = HG^2 - EG^2$.

110. PROBLEM.—*To make a parallelogram equivalent to a given square, and having the sum of its base*
F 73 *and altitude equal to a given line.* Let C (fig. 73) be the given square and A B the given line. SOL.—On A B as a diameter describe a semicircle. At A erect a perpendicular A D and make it equal to the side of the given square. Through D draw D E parallel to A B. From E let fall a perpendicular E F. Then A F will be the base and F B the altitude required, for they satisfy both conditions. *First* $AF \times FB = FE^2 = C$, that is the parallelogram is equivalent to the square. For (80) we have the proportion

$$AF : FE :: FE : FB,$$

whence (63) $AF \times FB = FE^2$. *Secondly* A F+F B=A B, that is the sum of the base and altitude is equal to the given line.

111. PROBLEM.—*To make a parallelogram equivalent to a given square, and having the difference of its*

base and altitude equal to a given line. Let the side of the given square be equal to A D (fig. 74) and let the given line be A B. Sol.—Place these so as to make a right angle at A and describe a circle upon A B as a diameter. Through D and O draw D F. Then D F will be the base and D E the altitude required, for they satisfy both conditions. *First* D F$\times$D E$=$D A^2, that is, the parallelogram is equivalent to the square. For (81) we have the proportion F 74

$$DF : DA :: DA : DE,$$

whence (63) D F$\times$D E$=$D A^2. *Secondly* D F—D E$=$E F$=$A B, that is the difference of the base and altitude is equal to the given line.

112. Problem.—*To make a square which shall be to a given square in any given ratio.* Let G H (fig. 75) be the side of the given square, and suppose it is required to make a square which shall be to the given square in the ratio of 3 to 7. Sol.—Make D C $=3$ and B D$=7$. On B C as a diameter describe a semicircle. At D erect the perpendicular D A. Through A and B draw A E$=$G H. Through E draw E F parallel to B C. Through A and C draw A C to meet E F. Then A F will be the side of the square required. We are to prove that F 75

$$AF^2 : AE^2 :: 3 : 7.$$

Now (70) we have

$$AF : AE :: AC : AB,$$

whence (67)

$$AF^2 : AE^2 :: AC^2 : AB^2.$$

It will be sufficient then to show that

$$AC^2 : AB^2 :: 3 : 7.$$

For this purpose we recur to (fig. 71). Since A B^2$=$ B E (108) and A C^2$=$C E, we have F 71

$$AB^2 : AC^2 :: BE : CE.$$

But B E$=$B F$\times$B D and C E$=$B F$\times$C D. Hence

$$AB^2 : AC^2 :: BF\times BD : BF\times CD.$$

Leaving out the common factor B F we have

$$AB^2 : AC^2 :: BD : CD,$$

that is—*the squares of the sides of a right triangle are to each other as the adjacent segments of the hypothenuse.* Therefore (fig. 75) F 75

$$AC^2 : AB^2 :: 3 : 7.$$

Consequently $AF^2 : AE^2 :: 3 : 7,$

which was to be demonstrated. The process would be the same for any other numbers instead of 3 and 7.

113. PROBLEM.—*To find the approximate ratio of the circumference of a circle to its radius or diameter.* This problem, on account of its vast practical importance, has received a variety of solutions. We propose the following as simpler than any we have met with. We have already shown (8) how to find the ratio of two straight lines; but it is obvious from the definition of a curve (10), that we cannot, in the same manner, find the *exact* ratio between a curve and a straight line, since we cannot find an *infinitely small* common measure. We can, however, approximate to perfect accuracy, just in proportion to the smallness of the common measure which we make use of. All this will be evident from what follows. SOL.—Let B C
F 76 (fig. 76) be the side of an inscribed hexagon. Then B C=the radius A B (92.) Now if we take B C as a common measure of the circumference and radius, it is contained 6 times in the circumference and 1 time in the radius. Accordingly our *first* approximate ratio is that of 6 to 1. This cannot be very near the truth, because the chord B C differs very perceptibly from the arc B D C. We shall therefore seek a smaller common measure. For this purpose we draw A D perpendicular to the middle of the chord B C, and it will bisect the arc B C in D (28). B D will then be the chord of $\frac{1}{12}$ of the circumference, and its value is found as follows. Calling A B=1 we have B E=$\frac{1}{2}$. Then (108) A E=$(\text{AB}^2-\text{BE}^2)^{\frac{1}{2}}=(1-\frac{1}{4})^{\frac{1}{2}}=0.866$. Knowing A E we have D E=A D—A E=1—0.866=0.134. Then in the right triangle B D E, we have B D=$(\text{BE}^2+\text{DE}^2)^{\frac{1}{2}}=(0.25+0.0179)^{\frac{1}{2}}=0.5176$. Now since B D is contained 12 times in the circumference, we have $12\times0.5176=6.2112$ for the circumference when the radius is 1. This is the *second* approximate ratio, and is much nearer the truth than the first, because the chord B D differs much less from the arc B D than the chord B C did from its arc. In order to make the third approximation, we draw A F perpendicular to the middle of the chord B D, and it will bisect the arc

B D in F. Then the chord B F is the chord of $\frac{1}{24}$ of the circumference, and its value is found in the same manner as that of B D. Thus A B$=$1 and B G$=\frac{1}{2}$ of B D$=$0.2588. Then A G$=$(A B^2—B G^2)$^{\frac{1}{2}}$$=$ (1—0.0669)$^{\frac{1}{2}}$$=$0.966. Knowing A G we have F G $=$A F—A G$=$1—0.966$=$0.034. Then in the right triangle B F G, we have B F$=$(B G^2$+$F G^2)$^{\frac{1}{2}}$$=$ (0.0669$+$0.00115)$^{\frac{1}{2}}$$=$0.2609. Now since B F is contained 24 times in the circumference, we have 24$\times$ 0.2609—6.2616 for the value of the circumference when the radius is 1. This is the *third* approximate ratio of the circumference to the radius, but it is still too small because the chord B F is still too large to be confounded with its arc, or to be considered as an *exact* common measure between the circumference and radius. It is obvious, however, that the process now commenced may be carried on to any limit we please, and each approximate ratio will be nearer the truth than the preceding, because, each arc being half the preceding, its chord, calculated exactly in the same manner as above, will constantly approach nearer and nearer to a coincidence with its arc. We shall not give the details of the calculation any further, because the preceding steps are sufficient to make the whole process intelligible. We shall only add that at the thirteenth division, where the arc is $\frac{1}{49152}$ of the circumference, and where the chord may be considered as almost exactly equal to its arc, the *fourteenth* approximate ratio is that of 6.2831852 to 1. If we call the diameter 1 the expression for the circumference is half the above, namely 3.1415926. Some persons have had the patience to extend the calculation to one hundred and forty decimals, but the above value is sufficiently accurate for all the purposes to which it ever becomes necessary to apply it. SCHO.—*Whenever the diameter of a circle is known, we obtain the circumference with sufficient accuracy by multiplying the diameter by* 3.1415926—*we obtain the diameter with sufficient accuracy by dividing the circumference by* 3.1415926,—*and we obtain the area with sufficient accuracy by multiplying the square of the radius by* 3.1415926. We have already seen (105) that the

area of a circle is equal to the circumference multiplied by half the radius, or $C \times \frac{1}{2}$ of R. But $C=2 R \times 3.1415926$. Hence the area—$2 R \times 3.1415926 \times \frac{1}{2}$ of $R = R^2 \times 3.1415926$. It is the practice of geometers to represent the above ratio 3.1415926 by the Greek character π, which is called *pi*. Then the expression for the circumference is $\pi \times D$, and for the area $\pi \times R^2$.

114. Problem.—*To make a square equivalent to any given figure.* This general problem is sometimes enunciated thus—*to find the quadrature of any given figure.* Sol.—In explaining its solution it will not be necessary to have recourse to a diagram. All the figures whose properties we have considered, except irregular polygons, are measured by a product consisting of two factors; and we have seen (107) that irregular polygons may be converted into equivalent triangles, and then the same will be true of them. Accordingly—*to make a square equivalent to any given figure, it is only necessary to find a mean proportional between the two factors by which that figure is measured.* This can always be done by the process before explained (80), and the mean proportional thus found will be the side of the square required; for the two factors will then be the extremes of a proportion, and the square of a mean proportional is always equal to the product of the extremes. Thus if the given figure be a parallelogram, find a mean proportional between the base and altitude (101). If a triangle, between the base and half the altitude (102). If a trapezoid, between the altitude and half the sum of the parallel sides (103). If a regular polygon, between the perimeter and half the radius of the inscribed circle (104). If a circle, between the circumference and half the radius (105). If a sector, between the arc and half the radius (106). If an irregular polygon, between the base and half the altitude of the equivalent triangle (107).

COMPARISON OF SURFACES.

115. Axiom.—*Any two surfaces are to each other as the products which express their areas.* Cor.—*When the two products have one factor the same, it*

may be left out (66), *and then the two surfaces will be to each other as the factors which are unlike. Thus if two parallelograms have the same altitude, they are to each other as their bases; if they have the same base, they are to each other as their altitudes—If two triangles have the same base, they are to each other as their altitudes; if they have the same altitude, they are to each other as their bases—If two trapezoids have the same altitude, they are to each other as the sums of their parallel sides; if the sums of their parallel sides are the same, they are to each other as their altitudes.*

SCHO.—The above proposition is not confined to figures of the same name. Any two figures may be compared by comparing their areas. Thus a triangle is to a circle, as the product of the base by half the altitude of the triangle, is to the product of the circumference by half the radius of the circle. And if the altitude of the triangle be equal to the radius of the circle, then the triangle is to the circle, as the base of the triangle is to the circumference of the circle.

116. THEOREM.—*Any two similar figures are to each other as the squares of their homologous sides.*

DEM. 1.—We begin with two similar triangles. Let these be A B C and D E F (fig. 77), and let A G F 77
be the altitude of the first and D H that of the second. By the preceding proposition, G

$$ABC : DEF :: BC \times AC : EF \times DH.$$

But from the similar triangles A B C and D E F we have (78) B C : E F :: A B : D E.

Moreover the triangles A B G and D E H are similar, since the angle B=E and they have each a right angle.

Hence A B : D E :: A G : D H.

The two last proportions have a common ratio A B : D E. Therefore

$$BC : EF :: AG : DH.$$

Multiplying this, term by term, (67) by the identical proportion B C : E F :: B C : E F, we have

$$BC^2 : EF^2 :: BC \times AG : EF \times DH.$$

But we had above

$$ABC : DEF :: BC \times AG : EF \times DH.$$

Leaving out the common ratio in the two last proportions, we have

$$ABC : DEF :: BC^2 : EF^2,$$

which was first to be demonstrated. 2. It will now be easy to generalize the demonstration for similar figures of any number of sides. Take the two similar
F 58 polygons A B C D E and F G H I K (fig. 58). These are composed of the same number of similar triangles (87). Hence A B C : F G H :: AC^2 : FH^2.
Also A C D : F H I :: AC^2 : FH^2.
Therefore A B C : F G H :: A C D : F H I.
In like manner

A C D : F H I :: A D E : F I K,

and so on for any number of triangles. Thus we have the continued proportion

ABC : FGH :: ACD : FHI :: ADE : FIK.

Here the sum of the antecedents is the polygon A B C D E, and the sum of the consequents is the polygon F G H I K. Therefore (69)

ABCDE : FGHIK :: ABC : FGH.

But ABC : FGH :: AB^2 : FG^2.
Therefore

ABCDE : FGHIK :: AB^2 : FG^2.

In other words, similar figures are to each other as the squares of their homologous sides.

117. Theorem.—*Circles are to each other as the squares of their radii.* No diagram is necessary for this demonstration. Let us call one circle A, its circumference C, and its radius R; and let us call another circle *a*, its circumference *c*, and its radius *r*. Dem.—Since surfaces are as their areas, we have

$$A : a :: C \times \tfrac{1}{2} R : c \times \tfrac{1}{2} r.$$

Doubling the second ratio,

$$A : a :: C \times R : c \times r.$$

But (96) $C : c :: R : r.$
Multiplying this last, term by term, by the identical proportion $R : r :: R : r$, we have

$$C \times R : c \times r :: R^2 : r^2.$$

Now we had before $A : a :: C \times R : c \times r.$
Hence, leaving out the common ratio,

$$A : a :: R^2 : r^2,$$

that is, circles are as the square of their radii, which was to be demonstrated.

118. Theorem.—*Equal perimeters do not always enclose equal areas.* This may be demonstrated by num ers without a diagram. Dem.—Take, for instance, a square and an oblong of equal perimeters.

Let the side of the square be 12 feet; then its perimeter is $12+12+12+12=48$ feet. Let the base of the oblong be 16 feet and its altitude 8 feet; Then its perimeter is $16+16+8+8=48$ feet. Thus the perimeters are equal. But the area of the square is $12\times12=144$ square feet; and the area of the oblong is $16\times8=128$ square feet. Therefore the areas are unequal.

119. Theorem.—*If two triangles have the same base and equal perimeters, that is the greatest in which the two undetermined sides are equal.* Let the two triangles be A C B and A M B (fig. 78) having the same base A B and equal perimeters. We say that if A C=C B and if A M is not equal to M B, then A C B is greater than A M B. Dem.—Produce A C till C D=C B. Join D and B and produce the line indefinitely. The angle A B D is a right angle, because if a circle were described with the centre C and radius C B, the angle A B D would be inscribed in a semicircumference. Now take M as a centre, and with a radius M B make an arc cutting D B produced in N, so that M N=M B. Then A M+M N=A M+M B=A C+C B=A C+C D=A D. But A M+M N is greater than A N, since A N is the shortest distance between A and N. Therefore A D is greater than A N. Now of two unequal oblique lines, that which is greater must be more distant from the perpendicular (31). Hence D B is greater than B N, and G B, half of D B, is greater than B P, half of B N. But G B is the altitude of A C B, and B P is the altitude of A M B. Accordingly, since triangles of the same base are to each other as their altitudes, and since G B is greater than B P, the triangle A C B is greater than A M B, which was to be demonstrated. F 78

120. Theorem.—*Among polygons of the same perimeter and the same number of sides, that is the greatest in which the sides are equal.* Let there be the polygon A B C D E F (fig. 79). If the sides are not equal, the polygon may be enlarged without enlarging the perimeter. Dem.—Draw B D. Then, by the preceding proposition, if B C is not equal to C D, the isosceles triangle B O D of the same base and perimeter is greater than B C D, and by substituting it, the polygon would be enlarged without enlarging its perim- F 79

eter. The same might be proved with respect to all the other sides. Therefore the greatest polygon, of a given perimeter and a given number of sides, must be that in which all the sides are equal, which was to be demonstrated.

121. THEOREM.—*Among polygons of the same perimeter and the same number of sides, the regular polygon is the greatest.* We have just proved that the sides must be equal, and we shall now prove that the angles must be equal. As the demonstration is long, we shall divide it into three distinct propositions. DEM. 1.—*Among triangles formed with two given sides, the greatest is that in which the two given sides make a right angle.* Let there be two triangles B A C and
F 80 B A D (fig. 80), having the side A B common, and A C=A D. Then, if B A C is a right angle, we say the triangle B A C is greater than the triangle B A D, in which the angle at A is not a right angle. For, since the triangles have the same base A B they are to each other as their altitudes A C and D E. But A C is greater than D E, since its equal A D is greater than D E (30). 2—*Among polygons in which all the sides but one are given, that is the greatest, of which all the given sides can be inscribed in a semicircle having the unknown side for its diameter.* Let the polygon
F 81 A B C D E F (fig. 81) be the greatest that can be made of the given sides, A B, B C, C D, D E, E F, and the unknown side A F. Then we say that the angle formed by drawing lines from any vertex as D to the extremities of A F, is a right angle, and consequently inscribed in a semicircumference. For if A D F is not a right angle, then by the preceding proposition, the portion A D F may be enlarged without altering the portions A B C D and D E F; and thus the polygon itself would be enlarged. But by supposition it is already the greatest possible. Therefore the angle A D F is a right angle, and the same might be proved of all the other vertices. Consequently A F is the diameter of a semicircle in which the given sides are inscribed. 3.—*Among polygons formed of given sides, the greatest is that which can be inscribed in a circle.* Let the polygon A B C D E F G (fig. 82) be inscribed; and let the polygon *a b c d e f g*, of which the sides are respectively equal to the former, be one

which cannot be inscribed. Then we say the former is the greatest. Draw the diameter E M and join A M and M B. Then upon $a\ b$=A B make the triangle $a\ b\ m$=A B M, and join $e\ m$. Now, according to the preceding proposition, the polygon E F G A M is greater than $e\ f\ g\ a\ m$, unless this last can be inscribed in a semicircle having $e\ m$ for its diameter, which by supposition cannot be done. For the same reason E D C B M is greater than $e\ d\ c\ b\ m$. Hence the entire polygon E F G A M B C D E is greater than $e\ f\ g\ a\ m\ b\ c\ d\ e$. Then subtracting the equal triangles A B M and $a\ b\ m$, we have the inscribed polygon A B C D E F G greater than the polygon $a\ b\ c\ d\ e\ f\ g$ which cannot be inscribed. We have now proved that the greatest polygon that can be formed with a given perimeter and a given number of sides, must be equilateral and capable of being inscribed in a circle. Then it must be regular; for it is equiangular as well as equilateral, since each of the inscribed angles has the same measure.

122. THEOREM.—*Among regular polygons of the same perimeter but a different number of sides, that is the greatest which has the greatest number of sides; and a circle is greater than any polygon of the same perimeter.* We shall need no diagram for this demonstration. DEM.—We have already seen (104) that the area of a regular polygon is equal to its perimeter multiplied by half the radius of the inscribed circle. Consequently any two regular polygons are to each other as the products of their perimeters by half the radii of the inscribed circles. But in the case before us, the perimeters being the same, they are common factors. Therefore the two polygons are as the half radii or (66) as the radii of the inscribed circles. But the radii are as their circumferences (96). It only remains then to prove that of two circles inscribed in regular polygons of equal perimeters, that is the greater which is inscribed in the polygon of the greater number of sides. Now the inscribed circle is always less than the polygon unless the number of sides is infinite. This is self-evident. It is equally evident, from mere inspection, that the difference becomes less as the number of sides of the polygon becomes greater. But by supposition we do not increase the perimeter by increasing the

number of sides. Accordingly the limit remains always the same, and that circle must be the greatest in which the difference between it and the polygon is least. But this, as we have just seen, is when the number of sides is greatest, which was to be proved. It follows, moreover, that a circle is greater than any polygon of the same perimeter, because here the number of sides is infinite.

PLANES AND THEIR ANGLES.

123. SCHO.—We have hitherto considered planes as bounded by lines enclosing determinate areas. We are now to consider them as occupying certain relative positions with respect to each other. In this view, they are to be no longer limited in extent, but to be regarded as indefinitely produced. And if we give them a determinate form in the diagrams, it is only because we cannot represent any other to the eye.

124. THEOREM.—*Two straight lines meeting each other, determine the position of a plane.* By this we mean that a plane can have but one position in which the two straight lines will lie in its surface. Let the two lines be A B and C B (fig. 83). DEM.—Let a plane be supposed to pass through A B, that is, to have A B in its surface. Then suppose this plane to turn about A B until the point C is in its surface. The line C B will be in its surface because two of its points are, and the position of the plane will be determined. For it is evident that it cannot be turned about either of the lines without departing from the other. COR.—*Three points not in the same straight line determine the position of a plane.* For they may be joined by two straight lines, which we have just shown will determine the position of a plane.

125. THEOREM.—*The intersection of two planes is a straight line* DEM.—It is a line because the planes have no thickness. And it is a straight line, because by the preceding proposition, if the points common to the two were not in the same straight line, the planes would have the same position, and consequently could not intersect each other.

126. THEOREM.—*The angle made by two planes is measured by the angle made by two lines, drawn perpendicular to the line of intersection, at the same point.* Two planes which intersect each other, must form an opening of greater or less extent. This opening is called a *plane angle.* Thus the opening made by the planes A B C D and A B E F which, for the sake of brevity, we shall call A C and A E (fig. 84), is a plane angle. We shall have a definite idea of this angle, if we suppose the plane A C at first to coincide with A E and then to turn about A B till it reaches its present position. DEM.—Suppose H I, in the plane A E, perpendicular to A B, and H G to have coincided with H I, while the planes coincided. Then during the motion of the plane A C, the point G would describe the arc G I. It is evident, moreover, that G I increases in the same proportion as the plane angle, since both are formed by one and the same motion. Therefore the arc G I may be taken for the measure of the plane angle. But the arc G I measures the linear angle G H I, formed by the two lines G H and I H perpendicular to A B. SCHO.—If the angle G H I is acute, or less than 90° the plane angle will be acute. If G H I is a right angle, the plane angle will be a right angle and the planes will be *perpendicular.* If G H I is obtuse, the plane angle will be obtuse. When the plane angle is either acute or obtuse, the two planes are said to be *oblique* or inclined to each other. Finally, since plane angles are measured by linear angles, it is evident that they have the properties of linear angles (20, 21, 22, 34, 35, 39). F 84

127. THEOREM.—*A straight line is perpendicular to a plane when it is perpendicular to two straight lines drawn through its foot in that plane.* A line is said to be perpendicular to a plane, when it is perpendicular to all the straight lines that can be drawn through its foot in the plane. For then it will make a right angle with the plane in every direction. DEM.—Let the plane be A C (fig. 85), and suppose E F perpendicular to F B and F C. We say that E F will be perpendicular to every other line F G drawn through its foot in the plane A C. Suppose F B to turn about E F remaining all the time perpendicular to it. Then one of its positions will be F C, since F C is by hypothesis perpendicular to E F. F 85

Now F B by its motion generates a plane to which E F is perpendicular, since it is perpendicular to the generating line in every position. This plane is B F C. But the plane B F C coincides with the plane A C (124), since by hypothesis the two planes have their position determined by the two lines F B and F C common to both. Hence the line F G is in the plane B F C. But E F is perpendicular to every line drawn through its foot in this plane. Therefore E F is perpendicular to F G. The same might be proved of any other line, since F G was taken at pleasure. Consequently E F is perpendicular to the plane A C, which was to be demonstrated.

128. THEOREM.—*A perpendicular measures the shortest distance from a point to a plane. Two oblique lines drawn equally distant from the perpendicular are equal—Of two oblique lines at unequal distances, the*
F 86 *more distant is the greater.* Let E F (fig. 86), be perpendicular to the plane A C, let E G and E H be two oblique lines drawn equally distant from the perpendicular, and let E I be drawn more distant than E G or E H. DEM. 1.—We say that E F is shorter than any other line. For E F is the side of a right triangle of which any other line as E G is the hypothenuse. 2.—We say that E G=E H. For the triangle E F G=E F H, having the two sides, E F, F G, and the included angle E F G respectively equal to the two sides E F, F H and the included angle E F H. Therefore E G=E H. 3.—We say that E I is greater than E G or E H. E I and E H are drawn to the same straight line and E I is more remote than E H. Therefore (31) E I is greater than E H, and consequently greater than any other line as E G drawn at the same distance as E H.

129. THEOREM.—*Two parallel lines comprehended between two parallel planes are equal.* Two planes are parallel when they are throughout at the same distance from one another. We have just proved that the shortest distance from a point to a plane is a perpendicular. Therefore—*two planes are parallel, or a straight line is parallel to a plane, when all the perpendiculars let fall from points in one to the other are equal.* This being admitted, we proceed to demonstrate the above proposition. DEM.—Let the two parallel planes be
F 87 A B and C D (fig. 87). If the two parallels are per-

pendicular to the planes the proposition is evident from the definition. But suppose they are not. Still we say they are equal. Let E F and G H be the two parallels oblique to the two planes. From E let fall the perpendicular E I to the plane C D, and from G let fall the perpendicular G K. Then the triangles F E I and H G K are equal. For the perpendicular E I= G K by definition, the angle I=the angle K being right angles, and the angle E=the angle G having their sides parallel and directed the same way. Therefore (55) the two triangles are equal, and E F=G H.

130. THEOREM.—*The intersections of two parallel planes by a third are parallel lines.* Let A B and C D (fig. 88) be two parallel planes intersected by a third plane H F, and let H G and E F be the intersections. Then we say that H G is parallel to E F. F 88
DEM.—In the plane H F draw the parallel lines H E and G F. These are equal by the preceding proposition. Then join E G. The triangles H E G and E G F are equal. For H E=G F, E G is common, and the angle H E G=the angle E G F being alternate-internal angles. Therefore (53) the triangles are equal, and the angle H G E=the angle G E F. Then these must be alternate-internal angles, and H G is parallel to E F (37).

131. THEOREM.—*Two straight lines comprehended between three parallel planes are divided proportionally.* Let the three planes be A B, C D, E F (fig. 89). F 89
DEM. 1.—Suppose the two lines to meet as G H and G I. By the preceding proposition the plane G H I will make the intersections K L and H I parallel. Then (70) G K : K H :: G L : L I.

2.—Suppose the two lines do not meet as G H and M I. Still we shall have

G K : K H :: M N : N I.

For by drawing G I, the plane G I M makes the intersections L N and G M parallel. Then (70)

G L : L I :: M N : N I.

But we had G K : K H :: G L : L I.

Hence, leaving out the common ratio G L : L I we have G K : K H :: M N : N I.

SECTION THIRD.

SOLIDS.

132. Def.—*A solid is that magnitude which has the three dimensions of extension, namely, length, breadth and thickness; and we may conceive it to be generated by the motion of a surface in any direction but that of its length or breadth.* Thus we have the origin of the three dimensions: for the moving surface has two, length and breadth, and the motion produces a third, namely, thickness. We have seen that points are the boundaries of lines, and lines the boundaries of surfaces. In like manner *surfaces are the boundaries of solids.* These surfaces may be either plane or curved, and the solids enclosed by them will have different denominations and properties accordingly.

133. Def.—The general name for solids bounded by planes is *polyedron.* The planes are called *faces,* and their lines of intersection *edges* or *sides.* The least number of planes which can enclose a space or bound a solid, is four. Three planes meeting each other, would make an opening called a *solid angle,* and a fourth is necessary to close up this opening. Thus
F 90 the three planes B A C, B A D, C A D (fig. 90), which meet in A, form an opening or solid angle at A, and a fourth plane B C D is necessary to close up this opening. The points A, B, C, D are called *vertices.* The solid A B C D is called, from the number of its faces, a *tetraedron.* For the same reason a solid of six faces is called a *hexaedron,* one of eight, an *octaedron,* and so on. But other denominations, depending upon the form and relative positions of the faces, are more important. *A prism is a solid comprehended under several parallelograms which terminate in two equal and parallel polygons.* Thus if the polygon A B C D E
F 91 (fig. 91) is equal and parallel to the polygon F G H I K, and if all the other faces are parallelograms, as they evidently must be (129, 130), then the solid A H is a prism. The two equal and parallel polygons are called the *bases* of the prism, and the sum of the parallelograms A F G B, B G H C, &c. are called the *convex*

surface of the prism. If the faces are perpendicular to the bases the prism is called a *right prism.* The *altitude* of a prism is a perpendicular let fall from one base to the other. If the bases of a prism be triangles, the prism is said to be *triangular;* if quadrilaterals, *quadrangular,* and so on. If the bases of a prism be parallelograms, then all the faces will be parallelograms, and the prism is called a *parallelopiped.* Thus A G (fig. 92) is a parallelopiped. If all the plane and linear angles are right angles, the prism is called a *right parallelopiped.* Among right parallelopipeds the *cube* is most remarkable, being comprehended under six equal squares. The only remaining polyedron to be mentioned is the *pyramid.* *A pyramid is a solid comprehended under several triangles proceeding from the same point and terminating in the sides of a polygon.* Thus A--B C D E F (fig. 93) is a pyramid. The point A is called the *vertex,* and the polygon B C D E F the *base.* The *altitude* of a pyramid is a perpendicular let fall from the vertex to the base. The sum of the triangles form the *convex surface* of the pyramid. If a plane as G H I K L pass through the pyramid parallel to the base, the part cut off below is called a *frustum of a pyramid.* The sum of the trapezoids thus formed is the *convex surface* of the frustum. If the base of a pyramid is a regular polygon and if the altitude passes through the centre of the base, the pyramid is said to be *regular,* and the altitude is called the *axis* of the pyramid.

F 92

F 93

134. Def.—Of the solids terminated by curved surfaces, only three are considered in the elements of geometry, namely, the *cylinder,* the *cone,* and the *sphere,* which are usually denominated the *three round bodies, or the three solids of revolution.—If a right parallelogram be supposed to revolve about one of its sides as a fixed axis, the solid thus generated will be a cylinder.* Thus if the right parallelogram A B G H (fig. 94) be supposed to revolve about A B, the solid E G is a cylinder. The two equal and parallel circles described by the radii A H and B G, are called the *bases* of the cylinder, the axis A B the *altitude,* and the path described by H G, the *convex surface.—If a right triangle be supposed to revolve about one of its sides which include the right angle, the solid thus generated*

F 94

will be a cone. Thus if the right triangle S A D
F 95 (fig. 95) revolve about S A as an axis, the solid S--B D C E is a cone. The circle described by the revolution of A D is called the *base* the point S the *vertex*, and the path described by the hypothenuse S D, the *convex surface*. The axis S A is the *altitude*, and any line S B drawn from the vertex to the circumference of the base, is called the *side* of the cone. If a plane as F G H I pass through the cone parallel to the base, the part cut off below is called a *frustum of a cone*. —*If a semicircle be supposed to revolve about its diameter, the solid thus generated will be a sphere.* Thus if
F 96 the semicircle M A P (fig. 96) revolve about M P, the solid thus generated will be a sphere. M P, the diameter of the generating circle, is the *diameter* of the sphere, and C P the *radius*. From the manner in which the sphere is generated, it follows that *every point in the surface of a sphere is equally distant from the centre.* Also—*If a plane be made to pass through the sphere in any direction, the section will be a circle.* If the plane pass through the centre as M D P, this is evident, since every point in the curve M D P is equally distant from the centre C. In this case the circle is called a *great circle.* If the plane does not pass through the centre as E H G, still the curve E H G I is a circle. Suppose the plane in question to be perpendicular to the diameter of the generating circle. It is immaterial whether this diameter be considered as M P or A B. Let it be A B. Then the curve E H G I may be conceived to be traced by the motion of the point G. But G remains always at the same distance from H. Therefore it describes a circle of which H is the centre. Now in whatever direction we suppose a plane to pass, it is evident that a diameter may be drawn perpendicular to it, and that this may be considered as the diameter of the generating circle. Then, from the reasoning just made use of, the section will be a circle, Hence the proposition is universally true. In this case when the plane does not pass through the centre of the sphere, the circle is called a *small circle.* If two parallel planes pass through a sphere, or if one be a tangent to the sphere, that is, if it touch the sphere only in one point, while the other passes through it, in either case the portion of the sur-

face comprehended between the two parallel planes is called a *zone*. Thus the portions of the surface A—E H G I and E H G I—M D P F are zones, and the circular planes are called their *bases*. Also the portion of the sphere cut off by a single plane, or comprehended between two parallel planes, is called a *spherical segment*. Thus the solids A—E H G I and E H G I—M D P F are spherical segments, and the circular planes are their *bases*. The *altitude* of a zone or segment is the perpendicular drawn between its bases. While the semicircle A P B generates the sphere, the sector B C K generates a solid which is called a *spherical sector*.

SURFACE OF POLYEDRONS.

135. THEOREM.—*The convex surface of a right prism is equal to the product of the perimeter of the base by the altitude.* DEM.—By the definition (133) the convex surface is composed of parallelograms. Moreover, the prism being a right prism, the altitude G D of the prism (fig. 91), is the common altitude of F 91
all the parallelograms; and the sum of their bases F G+G H+H I+&c. is the perimeter of the base of the prism. Therefore, adding together their areas, we have for the convex surface, the measure above enunciated.

136. THEOREM.—*The convex surface of a regular pyramid is equal to the product of the perimeter of its base, by half the altitude of one of its triangles.* DEM.—By the definition (133) all the triangles forming the convex surface of a regular pyramid are equal. For their bases are equal, being sides of a regular polygon; and the other sides are equal, being oblique lines drawn at equal distances from the perpendicular, since the axis passes through the centre. But triangles which are equal and have equal bases, must have the same altitude. Therefore, the sum of their bases multiplied by half the common altitude, is the same as the perimeter of the base of the pyramid multiplied by half this altitude.

137. THEOREM.—*The convex surface of the frus-*

tum of a regular pyramid is equal to the product of half the perimeters of the two bases, by the altitude of one of the trapezoids. DEM.—By the definitions (133) the trapezoids forming the convex surface are all equal, being remainders after taking equal triangles from equal triangles. Moreover they have their parallel sides equal each to each, since the two polygons are regular. Therefore they must have a common altitude; and this multiplied by half the sum of their parallel sides, is the same as the measure above enunciated.

SOLIDITY OF POLYEDRONS.

138. SCHO.—In order to find the solidity of polyedrons, we must fix upon some known solid as *a unit of solidity*, and see how many times it is contained in the solid to be measured. Of all solids the cube is most regular and simple; and accordingly the same reasons which induced geometers to adopt the square as the unit of surface, have also induced them to adopt *the cube as the unit of solidity*. The cube is a solid comprehended under six equal squares, and consequently has all its three dimensions the same; in other words its length, breadth, and thickness are expressed by the same linear unit, and each of its faces is the square of that linear unit. Thus a cubic inch is an inch long, an inch broad, and an inch thick, and so of a cubic foot, a cubic yard, &c. The unit of solidity, as well as the unit of surface, depends upon the linear unit. It is a cubic inch, when the length, breadth, and thickness are expressed in inches, a cubic foot, when expressed in feet, and so on.

139. THEOREM.—*The solidity of a right parallelopiped is equal to the area of its base multiplied by its*
F 97 *altitude.* Let the right parallelopiped be E C (fig. 97), having the right parallelogram E H G F for its base and F B for its altitude. DEM.—Suppose E H to contain a given number of inches as 9, and E F a given number as 5. Then (100) E H G F will contain 45 squares. Now each of these squares may be made the base of a cube, whose three dimensions are an inch. Then the first layer will contain 45 of these cubes. And it is evident that there

will be as many such layers as there are inches in the altitude, since this layer only takes up one inch F I of the altitude. Let the number of inches in the altitude F B be 8. Then the whole number of cubes contained in the right parallelopiped is $8\times45=360$. Thus the measure of its solidity is 360 cubic inches. We have here made use of particular numbers, but this is only for the sake of being definite. It is evident that the same reasoning would apply to any other numbers. If the dimensions contained fractions of an inch, the proposition would still be true, as might be shown by reasoning similar to that employeed in art. 100. Hence we conclude universally that the solidity of a right parallelopiped is equal to the area of its base multiplied by its altitude, which is the same as the product of its three dimensions. Thus the solidity of E C expressed in lines$=$E H$\times$E F$\times$F B. Cor.—*The solidity of a cube is found by taking one of its sides three times as a factor.*. Thus if the right parallelopiped be a cube, then E H$=$E F$=$F B, and E H$\times$E F$\times$F B$=$E H^3. This explains the reason why the term cube is used to express the *third power* of any number.

140. Theorem.—*The solidity of any parallelopiped is equal to the area of its base multiplied by its altitude.* This will be evident if we prove that *any parallelopiped is equivalent to a right parallelopiped of the same base and altitude.* As the demonstration is long, we shall divide it into three distinct propositions. Dem. 1.—*If two parallelopipeds have the same inferior base, and their superior bases comprehended between the same parallel lines, they are equivalent.* Let the two parallelopipeds be E D and E M (fig 98) having F 98
the inferior base E F G H common, and their superior bases A B C D and I K L M comprehended between the same parallels A M and B L. The figure thus constructed contains two triangular prisms F B K—E A I and G C L—H D M. This will be true whether I K falls upon D C or upon either side of D C. Now we say that these two prisms are equal. The proof is by superposition. The triangle H D M$=$the triangle E A I, having their three sides respectively equal. Therefore the inferior bases will coincide. Moreover since D C corresponds in length and direction with A B, the point C will fall upon B. For the same rea-

son L will fall upon K, and G upon F. Thus all the vertices of one prism coincide with those of the other, and the two prisms fill the same space. Now if the left hand prism be taken from the entire solid, there will remain the parallelopiped E M; and if the right hand prism be taken from the entire solid, there will remain the parallelopiped E D. But if equals be taken from the same thing, equals will remain. Therefore the two parallelopipeds are equivalent. 2.—*Any two parallelopipeds of the same base and altitude are equivalent.*
F 99 Let the two parallelopipeds be A H and A M (fig. 99), having the same inferior base A B C D, and their superior bases E F G H and I K L M in the same plane, the altitudes being the same. We say that A H=A M. Produce F E, G H, L K, M I. Their intersections will form a parallelogram N O P Q= E F G H=I K L M=A B C D. Then N O P Q may be considered as the base of a third parallelopiped A Q. Now A Q=A M, by the preceding proposition. For the same reason A Q=A H. Consequently A H=A M of the same base and altitude, which was to be demonstrated. 3.—*Any parallelopiped may be changed into an equivalent right parallelopiped of the same base and altitude.* *First* suppose the base is a right parallelogram, but the faces not perpendicular
F 99 and let A M (fig. 99) be the given parallelopiped. At the points A, B, C, D, erect perpendiculars to meet the plane of I K L M. Then A H will be a right parallelopiped; and by the preceding proposition it is equivalent to A M. *Secondly* suppose the faces perpendicular, but the base not a right parallelogram; and let the given parallelopiped be A B C D—E F G H
F 100 (fig. 100). From B and C let fall the perpendiculars B I and C K upon A K. B C K I will be a right parallelogram. From I and K erect the perpendiculars I M and K L to the plane A B C D, and join F M and G L. Then B C K I—F G L M is a right parallelopiped. But since consistently with the definition of a prism, any face of a parallelopiped may be taken for a base, the two parallelopipeds A H and B L have the same base B C G F; and the same altitude, since the opposite bases are in the same plane. Therefore, by the preceding proposition, they are equivalent. Hence *any parallelopiped is equivalent to a right parallelopi-*

ped of the same base and altitude. Being equivalent they have the same solidity. Therefore *the solidity of any parallelopiped is equal to the area of its base multiplied by its altitude.*

141. THEOREM.—*The solidity of a triangular right prism is equal to the area of its base multiplied by its altitude.* DEM.—Let B H (fig. 92) be a right parallelopiped. Let a plane pass through the vertices, E, A, C, G. This plane will divide the right parallelopiped into two triangular right prisms. Call these A F and A H. They are prisms (133), because their bases are equal and parallel and their other faces are parallelograms. Now we say that the two prisms A F and A H are equal. Suppose them entirely detached from each other by the plane E A C G. Suppose A F removed from its present position and so placed that B C=D A shall fall upon D A, the point B falling on D, and the point C on A. Then B A will fall on D C, since the angle B=D (84) and B A=D C. Thus the lower bases will coincide. Moreover B F will coincide with D H, since otherwise there would be two perpendiculars at the same point D in the line A D. Then the plane A B F E will coincide with the plane D C G H, since their positions are determined by the same three points H, D, C, not in a straight line (124). For the same reason the plane B C G F will coincide with the plane D H E A, and E F G with G H E. Thus the two right prisms A F and A H are equal. Then each is half the parallelopiped, and must have half its measure. But the solidity of the parallelopiped =A B C D×A E (139). Therefore the solidity of the right triangular prism A G=half A B C D×A E= A B C×A E=the area of its base multiplied by its altitude.

F 92

142. THEOREM.—*The solidity of any triangular prism is equal to the area of its base multiplied by its altitude.* DEM.—Let A F (fig. 101) be an oblique parallelopiped. The plane A D F C divides it into two oblique triangular prisms, which we will call A F and A O. We are to prove that these two prisms are equivalent. Suppose two planes A G H M and D I K N perpendicular to A D. Then we shall have two right prisms, which we will call A I and A N. These are equivalent by the preceding proposition. Then

F 101

the prism A E will be equivalent to A O, if we prove that A E=A I and that A O=A N. First the prism A E=the prism A I. The portion comprehended between A B C and D I K is common to both; and by superposition it may be shown that the solid D I K F E =the solid A G H C B. Place D I K upon A G H and, by the definition of a prism, they will coincide. Moreover K F will coincide with H C in direction, because there cannot be two perpendiculars erected at the same point; and in length, because F C=K H each being equal to A D, and taking away K C which is common, we have K F=H C. Therefore the point F will fall on C. By the same reasoning E will fall on B. Then the two solids will coincide throughout, since their corresponding planes are determined by the same points. Now if to the part which is common A B C—D I K, we add the upper solid, we have the oblique prism A E, and if to the same common part we add the lower solid, equal to the upper, we have the right prism A I. Therefore A E=A I. By the same reasoning A O=A N. But A I=A N. Therefore A E=A O. Thus each of the oblique prisms is half of the oblique parallelopiped, and must have its measure; that is, (140) half its base into its altitude. But half its base A B C L is A B C or A C L the base of the prism (84), and the altitude is the same. Therefore every triangular prism has for the measure of its solidity, the area of its base multiplied by its altitude.

143. Theorem.—*The solidity of any prism whatever is equal to the area of its base multiplied by its altitude.* Dem.—Suppose we have the prism G D
F 91 (fig. 91), the base of which is a pentagon. By the planes A F H C and A F I D, it is divided into three triangular prisms; that is, into as many as the base has triangles. In the same manner every prism may be divided into as many triangular prisms as the base has sides minus two. Moreover each of these prisms, by the last proposition, has for its measure its base multiplied by its altitude. But the altitude is the same in all, and the sum of the triangular bases is equal to the base of the entire prism. Therefore the entire prism has for the measure of its solidity, the area of its base multiplied by its altitude.

144. Theorem.—*The solidity of a triagular*

pyramid is equal to a third of the area of its base multiplied by its altitude. This will be evident if we prove that *a triangular pyramid is a third part of a triangular prism of the same base and altitude;* for then it must have a third part of the measure of the prism. We shall divide the reasoning into four distinct propositions. DEM. 1.—*If a pyramid be cut by a plane parallel to the base, the section is a polygon similar to the base.* Let G H I K L (fig. 93) be a section F 93
parallel to the base B C D E F. We say the two polygons are similar. *First* their homologous sides are proportional. G H is parallel to B C, and H I to C D (130.) Then the triangle A G H is similar to A B C, and A H I to A C D (77). Hence.

A H : A C :: G H : B C

and A H : A C :: H I : C D.

Therefore (64) G H : B C :: H I : C D.

The same reasoning might be continued round the polygons. Therefore the homologous sides are proportional. *Secondly* the angles are equal each to each. G H I= B C D, if the triangles G H I and B C D are similar. We had the proportion

G H : B C :: H I : C D.

Also from the proportions

A G : A B :: G H : B C

and A G : A B :: G I : B D,

we have G H : B C :: G I : B D.

Therefore the triangles G H I and B C D are similar (76), and the angle G H I=B C D. In the same manner we might prove that H I K=C D E, and so of the rest. Therefore the polygons are similar. 2.—*If two pyramids have their bases in the same plane and equivalent, and the same altitude, the sections made by a plane parallel to the plane of the bases will also be equivalent.* Let the two pyramids be S-A B C and *s-a b c* (fig. 102), having their bases A B C F 102
and *a b c* in the same plane and equivalent; and having the same altitude, because their vertices S and *s* are in a line parallel to the plane of the bases. We say that the sections D E F and *d e f*, made by a plane parallel to that of the bases, are equivalent. By the preceding proposition, A B C is similar to D E F, and *a b c* to *d e f*. Therefore (116)

A B C : D E F :: BC^2 : EF^2

and $a\,b\,c : d\,e\,f :: b\,c^2 : e\,f^2$.
But (130) $B\,C : E\,F :: S\,C : S\,F$.
Hence (67) $B\,C^2 : E\,F^2 :: S\,C^2 : S\,F^2$.
Then $A\,B\,C : D\,E\,F :: S\,C^2 : S\,F^2$,
and in like manner

$$a\,b\,c : d\,e\,f :: s\,c^2 : s\,f^2$$

But (131) $S\,C : S\,F :: s\,c : s\,f$;
whence $S\,C^2 : S\,F^2 :: s\,c^2 : s\,f^2$.
From the four last, by leaving out equal ratios, we have

$$A\,B\,C : D\,E\,F :: a\,b\,c : d\,e\,f$$

or (65) $A\,B\,C : a\,b\,c :: D\,E\,F : d\,e\,f$.
But $A\,B\,C = a\,b\,c$; therefore $D\,E\,F = d\,e\,f$. In the same manner $G\,H\,I = g\,h\,i$, and so on. 3.—*Two triangular pyramids which have equivalent bases and equal altitudes are equivalent.* We say that the prism
F 102 S-A B C (fig. 102)$=s$–$a\,b\,c$. The method of proof is by the *reductio ad absurdum*, and was invented by Queret. It is as follows. If S-A B C is not equivalent to s–$a\,b\,c$, let s–$a\,b\,c$ be the less; and suppose the difference equal to a prism which has A B C for its base, and any line N T for its altitude. Divide the entire altitude N R into equal parts each less than N T, and let one of these parts be N O. Through the points of division, O, P, Q, let planes pass parallel to that of the bases, as in the figure. By the preceding proposition, the sections are equivalent, since $A\,B\,C = a\,b\,c$. Above the triangles A B C, D E F, &c. construct *exterior* prisms, and below the triangles, $d\,e\,f$, $g\,h\,i$, &c. construct *interior* prisms, as in the two figures. Now it is evident that the sum of the exterior prisms is greater than S–A B C, and that the sum of the interior prisms is less than s–$a\,b\,c$. Hence the difference between these two sums must be greater than the difference between the two pyramids. How the difference between the sums of the exterior and interior prisms, is equal to the prism which has A B C for its base and N O for its altitude. Why? Because the second exterior prism is equivalent to the first interior, having equivalent bases D E F and $d\,e\,f$, by the preceding proposition, and the same alitude; whence (142) they have the same solidity. In like manner, the third exterior prism is equivalent to the second interior, and so on. Thus all the exterior prisms but the lower one, have equivalent interior ones. Therefore the difference between them is

the lower prism, namely, that whose base is A B C and altitude N O. Now if our first supposition be correct, this last is greater than the prism, which has the same base A B C and a greater altitude N T; which is manifestly absurd. This absurdity arises from supposing the two pyramids to differ. Therefore we conclude that they are equivalent or have the same solidity. 4.—*A triangular pyramid is a third part of a triangular prism of the same base and altitude.* Let A F (fig. 103) be a triangular prism. By the plane E A C cut off the triangular pyramid E–A B C, of the F 103
same base and altitude of the prism. Then there will remain the quadrangular pyramid E–A C F D. Divide this by the plane D E C into two triangular pyramids E–A C D and E–C F D. Thus the prism is divided into three pyramids. We say these three pyramids are equivalent. E–D A C=E–D F C by the preceding proposition, since they have equal bases D A C, D F C (84), and the same altitude, since the perpendicular let fall from E is the altitude of both. Again E–D F C=E–A B C; for, instead of E–D F C we may change the vertex and say C–D E F. Then C–D E F=E–A B C, since they have equal bases, by the definition of a prism, and for their common altitude, the altitude of the prism. Thus the three pyramids are equivalent. Each, therefore, is one third of the prism and must have one third of its measure; that is, one third of the product of the base by the altitude, which was to be demonstrated.

145. Theorem.—*The solidity of any pyramid whatever is equal to a third of the area of its base multiplied by its altitude.* Dem.—For any pyramid as A–B C D E F (fig. 93), may be divided into tri- F 93
angular pyramids having the same altitude as the entire pyramid, and the sum of whose bases makes the base of the entire pyramid.

146. Theorem.—*The solidity of the frustum of any pyramid may be found by adding together the upper base, the lower base, and a mean proportional between the two bases, and multiplying the sum by a third of the altitude of the frustum.* Thus if A is the upper base, B the lower base, and H the altitude of the frustum, then the solidity$=(A+B+(AB)^{\frac{1}{2}}\times\frac{1}{3}H$. It

will be sufficient to demonstrate this with respect to
F 105 the triangular frustum A B C–D E F (fig. 105), if we first demonstrate that *any frustum is equivalent to a triangular frustum of equivalent bases and the same altitude.* DEM. 1.—Let there be two pyramids
F 104 A–B C D E F and G–H I K (fig. 104) of the same altitude. Let the bases be in the same plane and equivalent, and let a plane parallel to that of the bases make the sections *b c d e f* and *h i k*. These are equivalent (144). Thus the two frustums have the same altitude and equivalent bases. Then we say they are equivalent. For the entire pyramids are equivalent, since they have equivalent bases and the same altitude; and the partial pyramids A–*b c d e f* and G–*h i k* are equivalent for the same reason. Thus the two frustums are what remain after taking equal soliditics from equal solidities. Consequently the triangular frustum is equivalent to the other.
F 105 2.—The triangular frustum A B C–D E F (fig. 105) may be divided into three pyramids, having for their common altitude the altitude of the frustum, and for their respective base, the lower base of the frustum, the upper base, and a mean proportional between the two. The plane A E C cuts off one pyramid E–A B C, which has for its base the lower base of the frustum, and for its altitude, the altitude of the frustum. The plane D E C cuts off another pyramid C–E D F, which has for its base the upper base of the frustum, and for its altitude, the altitude of the frustum. There remains the pyramid E–D A C, for which we may substitute, G–D A C, by taking G in a line E G parallel to the base; for the two pyramids E–D A C and G–D A C have the same base and altitude and are therefore equivalent. But instead of G–A D C, we may take D for the vertex and A G C for the base. Thus we have a third pyramid D–A G C, which has for its altitude the altitude of the frustum. It only remains to prove that its base A G C is a mean proportional between A B C and D E F; in other words, that A B C : A G C : : A G C : D E F.
Now C being the common vertex of A B C and A G C, they have the same altitude. Therefore (115) they are to each other as their bases; that is

A B C : A G C : : A B : A G or D E.

But A B C and D E F being similar (144),

A B : D E : : A C : D F.

Therefore A B C : A G C : : A C : D F.

Again, since G E is by construction parallel to the plane in which A C and D F are situated, the triangles A G C and D E F have the same altitude. Therefore (115)

A G C : D E F : : A C : D F.

Then, from the two last proportions, leaving out the common ratio, we have

A B C : A G C : : A G C : D E F.

Whence A G C$=$(A B C$\times$D E F)$^{\frac{1}{2}}$. If now we add together the solidities of the three pyramids which compose the frustum, we shall have the result enunciated at the head of the article.

147. Theorem.—*The solidity of a truncated triangular prism is equal to that of three pyramids, having for their common base the base of the prism, and for their vertices, the three vertices of the inclined section.* If a prism be cut by a plane inclined to the base, the part cut off is called a *truncated prism.* Thus if D E F (fig. 103) is not parallel to the base F 103
A B C, this solid is a truncated prism. And we say that the truncated prism A B C–D E F$=$E–A B C$+$D–A B C$+$F–A B C. Dem.—The plane A E C cuts off the first E–A B C. Then E–A C F D remains. This is divided by the plane D E C into E–A D C and E–F D C. Now E–A D C$=$B–A D C, since they have the same base and altitude; and B–A D C is the same as D–A B C which forms the second pyramid above mentioned. Lastly E–F D C$=$D–E F C$=$A–E F C$=$E–A F C$=$B–A C F$=$F–A B C, which is the third pyramid enunciated. Cor.—*The solidity of a truncated prism is found by addiug together the altitudes of the three vertices of the inclined section, and multiplying their sum by one third of the area of the base.*

148. Scho.—The polyedrons whose solidity has now been ascertained, namely, the *prism*, the *pyramid*, the *frustum of a pyramid*, and the *truncated triangular pirism*, are the only ones, for the measurement of which specific rules can be given. If we have any other polyedron, its solidity must be obtained by dividing it into pyramids and measuring these pyramids

separately. This may be done by taking any vertex and making planes to pass through the edges meeting in this vertex. Then there will be as many pyramids as there are faces in the polyedron, minus those which have one point in the common vertex.

SURFACE OF THE THREE ROUND BODIES.

149. THEOREM.—*The convex surface of a cylinder is found by multiplying the circumference of the base by the altitude; and the entire surface of a cylinder is found by adding together its axis and the radius of its base, and multiplying their sum by the circumference of the base.* Call the axis A, the radius R, and the circumference C. Then we say the convex surface$=A\times C$, and the entire surface$=(A+R)\times C$. DEM. 1.—The cylinder is a right prism of an infinite number of faces, because its bases being circles, are polygons of an infinite number of sides. Thus in the
F 106 circle which forms the base of the cylinder (fig. 106), each one of these infinitely small sides is the base of a right parallelogram, which makes the cylinder a prism. It is, moreover, a right prism, because by the definition, the line which generates the convex surface, is perpendicular to the line which generates the base. But we have proved (135) that the convex surface of a right prism is equal to the perimeter of the base multiplied by the altitude. Consequently the convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude; but the altitude is the same as the axis. Therefore the convex surface$=$ $C\times A$. 2.—The entire surface is made up of the convex surface and the two bases which are equal circles. Now one base$=C\times\frac{1}{2}R$ (105). Then both bases$=C\times R$. Adding this to $C\times A$, we have for the entire surface $(C\times A)+(C\times R)=(A+R)\times C$.

150. THEOREM.—*The convex surface of a cone is found by multiplying the side by half the circumference of the base; and the surface of a cone is found by adding together the radius of the base and the side of the cone, and multiplying their sum by half the circumference of the base.* By the side of the cone
F 107 we mean A E (fig. 107), the hypothenuse of the generating triangle. Then, calling the circumference

C, the radius R, and the side S, we say the convex surface$=S\times\frac{1}{2}C$ and the entire surface$=(R+S)\times\frac{1}{2}C$. DEM. 1.—The cone is a regular pyramid of an infinite number of faces. For the base is a regular polygon of an infinite number of sides, and the axis or altitude passes through the centre; which are the two conditions required for a regular pyramid (133). But we have proved (136) that the convex surface of a regular pyramid is equal to the perimeter of its base multiplied by half the altitude of one of its triangles. Now in the case of a cone, the perimeter is C and the altitude S. Therefore the convex surface$=S\times\frac{1}{2}C$. 2.—The entire surface is made up of the convex surface and the base. But the base is a circle and therefore (105)$=R\times\frac{1}{2}C$. Then adding together the two, we have $(S\times\frac{1}{2}C)+(R\times\frac{1}{2}C)=(S+R)\times\frac{1}{2}C$.

151. THEOREM.—*The convex surface of the frustum of a cone is found by multiplying half the sum of the two circumferences by the side; and the entire surface of the frustum of a cone is found by adding the side to the greater radius and multiplying the sum by half the greater circumference; then by adding the side to the less radius and multiplying the sum by half the less circumference; and lastly by adding these two products together.* Thus, if S be the side of the frustum, R the radius of the greater base and C its circumference, r the radius of the less base, and c its circumference; then we say the convex surface$=\frac{1}{2}(C+c)\times S$; and the entire surface of the frustum$=\frac{1}{2}C\times(R+S)+\frac{1}{2}c\times(r+S)$. DEM. 1.—The frustum of a cone is the frustum of a regular pyramid of an infinite number of faces, because its bases, being circles, are polygons of an infinite number of sides. Therefore (137) the convex surface$=\frac{1}{2}(C+c)\times S$. This is the same as $(\frac{1}{2}S\times C)+(\frac{1}{2}S\times c)$. 2.—The entire surface is made up of the convex surface and the two bases. But the two bases are circles, of which the greater$=\frac{1}{2}C\times R$; and the less$=\frac{1}{2}c\times r$. Then adding all the areas together, we have $(\frac{1}{2}C\times R)+(\frac{1}{2}c\times r)+(\frac{1}{2}C+S)+(\frac{1}{2}c\times S)=\frac{1}{2}C\times(R+S)+\frac{1}{2}c\times(r+S)$.

152. THEOREM.—*The surface of a sphere is equal to the circumference of a great circle multiplied by its diameter.* By the definition of a sphere and of a great

circle (134) the revolving circle is a great circle and the axis of revolution is the diameter of the sphere. We shall divide the demonstration into three parts.

DEM. 1.—*If a straight line revolve about another straight line as an axis, the surface generated thereby is equal to the revolving line multiplied by the circumference described by its middle point.* *First*, if the
F 94 revolving line is parallel to the axis, as C D (fig. 94), the surface generated is the convex surface of a cylinder, which (149) is equal to the circumference of the base multiplied by the axis. But the circumference of the base is the same as the circumference described by the middle point, and the revolving line is equal to the axis. *Secondly*, if the revolving line meets the
F 95 axis, as S B (fig. 95), the surface generated is the convex surface of a cone, which (150) is equal to S B multiplied by half the circumference described by the radius A B; or by abbreviation S B$\times\frac{1}{2}$ circ. A B. Now suppose H the middle point of S B. Then we say that $\frac{1}{2}$ circ. A B=circ. K H. For

circ. A B : circ. K H : : A B : K H (96).

But A B : K H : : S B : S H : : 2 : 1 (78).

Therefore circ. A B : circ. K H : : 2 : 1;

that is, circ. K H=$\frac{1}{2}$ circ. A B; and S B$\times\frac{1}{2}$ circ. A B=S B$\times$circ. K H. *Thirdly*, if the revolving line is inclined to the axis without meeting it, as K D
F 108 (fig. 108) the surface generated is the convex surface of the frustum of a cone, which (151) is equal to K D$\times$ ($\frac{1}{2}$ circ. C D+$\frac{1}{2}$ circ. I K). Now let G be the middle point of K D. Then we say that circ. F G=$\frac{1}{2}$ circ. C D+$\frac{1}{2}$ circ. I K. Since the circumferences are to each other as their radii, it is sufficient to prove that F G=$\frac{1}{2}$ (C D+I K). Draw K M parallel to I C the axis of the frustum. Then F L=$\frac{1}{2}$ (C M+I K). Also L G=$\frac{1}{2}$ M D; for (78)

L G : M D : : K G : K D : : 1 : 2,

that is L G=$\frac{1}{2}$ M D. Then F L+L G=$\frac{1}{2}$ (C M+M D+I K), or F G=$\frac{1}{2}$ (C D+I K). Consequently circ. F G=$\frac{1}{2}$ circ. C D+$\frac{1}{2}$ circ. I K, and the surface generated by K D=K D$\times$circ. F G. Hence the proposition is universally true. 2.—*If a regular semi-polygon revolve about its axis, the surface generated is equal to the axis multiplied by the circumference of the*
F 109 *inscribed circle.* Thus if A B C D E F G (fig. 109)

revolve about A G as an axis, we say that the surface generated$=$A G$\times$circ. H I. Take any one of the sides as B C. By the preceding proposition, the surface generated by B C$=$B C$\times$circ. H I. But B C$\times$circ. M I $=$N L$\times$circ. M I. For the triangles B C K and H I M are similar (77) since the sides of the one are perpendicular to those of the other. Then

B C : B K : : H I : M I : : circ. H I : circ. M I. Hence (63) B C$\times$circ. M I$=$B K$\times$circ. H I. But B K$=$N L. Therefore the surface generated by B C$=$N L$\times$circ. H I. In like manner the surface generated by C D$=$L H$\times$circ. H I, and the same is true with respect to the surfaces generated by each of the other sides; that is, the surface generated by each side, is equal to the circumference of the inscribed circle multiplied by the segment of the axis comprehended between the perpendiculars let fall from the extremities of that side. Therefore the entire surface generated by all the sides, is equal to the circumference of the inscribed circle multiplied by the sum of all the segments or the entire axis. 3.—*The surface generated by the revolution of a semicircumference about its diameter, is equal to the diameter multiplied by the circumference.* This follows directly from the preceding proposition. For the semicircumference is a regular semi-polygon of an infinite number of sides, of which the axis A G is the diameter; and the circumference of the inscribed circle is the same as that of the revolving circle. For when B C becomes infinitely small, H I will not differ from H A; and consequently circ. H I will not differ from circ. H A. Now circ. H A is that of a great circle of the sphere whose diameter is A G. Therefore the surface of the sphere is equal to its diameter multiplied by the circumference of a great circle. Scho.—*How to express the surface of a sphere.* If we call π the ratio of the circumference to radius (113), R the radius and C the circumference of a great circle, we have C$=2\ \pi\times$R, and D$=2$ R. Therefore the surface of the sphere$=$D$\times$C$=4\ \pi\times R^2$.

153. Theorem.—*The surface of a zone is equal to its altitude multiplied by the circumference of a great circle.* By the definition of a zone (134) it is the portion of the surface of a sphere generated by any arc of the revolving semicircumference. Thus the arc C D

F 109 (fig. 109) generates a zone of two bases, namely, circ. H D and circ. L C. Also the arc A B generates a zone of one base, namely, circ. N B. Both surfaces have the measure enunciated. DEM.—For the arc C D may be considered as composed of straight lines, and then, by the preceding proposition, the surface generated will have for its measure L H$\times$circ. H I. But L H is the altitude of the zone, and circ. H I when the side upon which H I falls is infinitely small, becomes the circumference of a great circle. For the same reason the zone generated by A B is measured by A N $\times$circ. H A. SCHO.—*How to express the surface of a zone.* Call A the altitude and C the circumference of a great circle. Then $C\times A=2\,\pi\times R\times A$. Moreover if we wish to find the surface of a *spherical segment* (134), of which the zone forms the convex surface, we have only to add to the surface of the zone, the areas of the two circular bases, or that of the single base, as the case may be.

154. THEOREM.—*The surface of a sphere is equal to that of four great circles ; and the surface of an inscribed sphere is equal to two thirds of the surface of the circumscribed cylinder.* A sphere is said to be inscribed in a cylinder, when the bases and convex surface of the cylinder are tangents to the sphere. Thus
F 110 if the semi-square A B F E (fig. 110) and the semicircle F G E revolve about the same axis F E, the sphere will be inscribed in the cylinder. DEM. 1.—The area of a great circle is equal to the circumference multiplied by half the radius, and the surface of the sphere is equal to the same circumference multiplied by the diameter (152). But the diameter is four times one half the radius. Therefore the surface of a sphere is four times that of a great circle of the sphere. 2.—The convex surface of the cylinder is also equal to that of four great circles, for the circumference of the base multiplied by the altitude (149) is the same as that of a great circle by its diameter. If to this we add the two bases, which are great circles, we have the entire surface of the cylinder equal to that of six great circles. Consequently the two surfaces are to each other as 4 to 6 or as 2 to 3.

SOLIDITY OF THE THREE ROUND BODIES.

155. Theorem.—*The solidity of a cylinder is equal to the area of its base multiplied by its altitude.* Dem.—We have already seen (149) that the cylinder may be regarded as a prism of an infinite number of faces. Then its solidity must be measured in the same manner as that of a prism, namely, by multiplying the area of the base by the altitude (143). Scho.—*How to express the solidity of a cylinder.* Call the axis A and the radius R. Then the area of the base$=\pi\times R^2$ and the solidity$=\pi\times R^2\times A$.

156. Theorem.—*The solidity of a cone is equal to one third of the area of the base multiplied by the altitude.* Dem.—We have already seen (150) that the cone may be regarded as a pyramid of an infinite number of faces. Then its solidity must be measured in the same manner as that of a pyramid, namely, by multiplying one third of the area of the base by the altitude (145). Scho.—*How to express the solidity of a cone.* Call the altitude A, and π and R as before. Then the area of the base$=\pi\times R^2$ and the solidity$=\frac{1}{3}\,\pi\times R^2\times A$.

157. Theorem.—*The solidity of the frustum of a cone is found by adding together its greater base, its less base, and a mean proportional between them, and then multiplying their sum by one third of the altitude.* Dem.—We have already seen (151) that the frustum of a cone may be regarded as the frustum of a pyramid of an infinite number of faces. Then its solidity must be measured in the same manner, namely, by adding together the solidities of three cones having for their respective bases, the greater base, the less base, and a mean proportional between them, and for their common altitude the altitude of the frustum (146). Scho.—*How to express the solidity of the frustum of a cone.* If we call R the greater radius, r the less, and A the altitude, we have the greater base$=\pi+R^2$, the less base$=\pi\times r^2$, and the mean proportional between the two$=(\pi\times R^2\times\pi\times r^2)^{\frac{1}{2}}=\pi\times R\times r$. Then adding these three together, we have their sum$=\pi\,(R+{}^{2}r^2+$

$R \times r$). Lastly, multiplying by $\frac{1}{3}$ A, we have the solidity$=\frac{1}{3}\pi \times A \times (R^2+r^2+R \times r)$.

158. THEOREM.—*The solidity of a sphere is equal to its surface multiplied by a third of its radius.* DEM.—If we take the smallest portion of the surface of a sphere that can be conceived, it will not differ perceptibly from a plane. Accordingly *we may consider the surface of a sphere as composed of infinitely small planes*, (98). Then, each one of these planes being taken for the base of a pyramid whose vertex is at the centre of the sphere, we shall have the solidity of the sphere by adding together the solidities of these pyramids Now all the pyramids have for their common altitude the radius of the sphere; consequently the sum of their bases multiplied by one third of the common altitude, is the same as the surface of the sphere multiplied by a third of the radius ; the measure which was enunciated. SCHO.—*How to express the solidity of a sphere.* Call R the radius of the sphere. Then $\pi \times R^2$=area of a great circle (113,) and $4\pi \times R^2$= surface of the sphere (154). Multiplying this last by $\frac{1}{3}$ R, we have $\frac{4}{3}\pi \times R^3$ for the solidity of the sphere.

159. THEOREM.—*The solidity of a spherical sector is equal to the zone which forms its base multiplied by one third of the radius.* Let the spherical sector be that which is generated by the revolution of the circular sector F G H (fig. 111). DEM.—By the reasoning of the preceding proposition, the zone generated by the arc F G may be considered as composed of infinitely small planes, each forming the base of a pyramid whose altitude is the radius H G. Then the solidity of the sector will be equal to the sum of these bases multiplied by $\frac{1}{3}$ H G ; that is, equal to the zone which forms the base of the sector, multiplied by one third of the radius. SCHO.—*How to express the solidity of a spherical sector.* Call the altitude of the zone A. Then (153) the zone$=2\pi \times R \times A$. Multiplying this by $\frac{1}{3}$ R, we have the solidity$=\frac{2}{3}\pi \times R^2 \times A$.

160. THEOREM.—*The solidity of an inscribed sphere is equal to two thirds of that of the circumscribed cylinder.* DEM.—The base of the cylinder
F 110 (fig. 110) being equal to a great circle of the sphere (154), the solidity of the cylinder is equal to a great circle multiplied by the diameter. Now the solidity of

the sphere (158) is equal to four great circles multiplied by $\frac{1}{3}$ of the radius, or $\frac{1}{6}$ of the diameter, which is the same as a great circle multiplied by $\frac{4}{6}$ or $\frac{2}{3}$ of the diameter. Therefore the two solidities are to each other as 2 to 3.

161. THEOREM.—*The solidity of a spherical segment of one base, is found by taking the difference of the solidities of a spherical sector and cone, when the segment is less than a hemisphere; and the sum, when the segment is greater than a hemisphere.* DEM.—This will be evident from a mere inspection of the figure. 1.—The spherical segment generated by P F G (fig. 111) is equal to the sector generated by H F G *minus* F 111 the cone generated by H F P; or more briefly, segment P F G=sector H F G—cone H F P. 2.—The spherical segment generated by O E A is equal to the spherical sector generated by H E A *plus* the cone generated by H E O; or more briefly, segment O E A=sector H E A+cone H E O.

162. THEOREM.—*The solidity of a spherical segment of two bases is found by taking the difference between the solidities of two spherical segments, which have for their respective single bases, the two bases of the segment to be measured.* DEM.—This also will be evident from inspection of the figure. Thus the segment generated by O E F P (fig. 111) is equal to the F 111 difference between the segment generated by O E G and the segment generated by P F G. Now by the preceding proposition, segment O E G=sector H E G —cone H E O; and segment P F G=sector H F G—cone H F P. Therefore segment O E F P=sector H E G—sector H F G+cone H F P—cone H E O.

COMPARISON OF SOLIDS.

163. SCHO.—It is easy to compare solids after having ascertained the measures of their solidity ; since for this purpose it is only necessary to compare those measures. Moreover if, in comparing two solidities, there be a common factor it may be omitted. Nor is the comparison limited to solids of the same kind. A prism may be compared with a sphere, or a cone with the frustum of a pyramid, for their ratio must be the

same as that of their solidities. The following propositions, therefore, may be received as corollaries of the preceding demonstrations. COR.—*Two prisms, two pyramids, two cylinders, or two cones are to each other as the products of their bases by their altitudes. If the altitudes are the same, they are as their bases. If the bases are the same, they are as their altitudes.*

164. THEOREM.—*The surfaces of two spheres are to each other as the squares of their radii, and the solidities are as the cubes of their radii.* DEM. 1.—Let S be the surface of one sphere, C a great circle of that sphere, and R its radius: also let s be the surface of another sphere, c a great circle of that sphere, and r its radius. Then (154) $S=4\,C$, and $s=4\,c$. But (117)

$$C : c :: R^2 : r^2,$$

and (66) $$4\,C : 4\,c :: R^2 : r^2;$$

whence $$S : s :: R^2 : r^2,$$

that is, the surfaces are as the squares of their radii. 2.—The solidities of the two spheres are to each other as their surfaces multiplied by one third of their radii (158); that is, as $S \times \frac{1}{3}R$ is to $s \times \frac{1}{3}r$. But since

$$S : s :: R^2 : r^2,$$

we have (65, 66,)

$$S \times \tfrac{1}{3}R : s \times \tfrac{1}{3}r :: \tfrac{1}{3}R^3 : \tfrac{1}{3}r^3 :: R^3 : r^3;$$

that is, the solidities of the two spheres are as the cubes of their radii.

SIMILAR SOLIDS.

165. DEF.—*Two polyedrons of the same number of faces are similar, when their homologous solid angles are equal, and their homologous faces are similar polygons. Also two cones or two cylinders are similar, when their altitudes are to each other as the radii of their bases.* The following are corollaries of the first definition. COR.—*The homologous sides or edges of similar polyedrons are proportional; and the homologous faces are to each other as the squares of their homologous sides.*

166. THEOREM.—*Two similar pyramids are to each other as the cubes of their homologous sides.* DEM.—Since by the definition the homologous solid angles are equal, the less pyramid may be placed in the
F 93 greater so that the solid angles at A (fig. 93) shall coin-

cide. Moreover since the base G H I K L is, by the definition, similar to B C D E F, and is at the same time a section of the greater pyramid, the two bases are parallel (144). Now calling A F and A L the altitudes, (163)

A–B C D E F : A–G H I K L : : B C D E F$\times$ A F : G H I K L$\times$A L.

But

B C D E F : G H I K L : FE^2 : LK^2

and (131) A F : A L : : F E : L K.

Multiplying these two proportions term by term (67) we have

B C D E F$\times$A F : G H I K L$\times$A L : : FE^3 : LK^3.

Therefore, substituting this last ratio for its equal in the first proportion, we have

A–B C D E F : A–G H I K L : : FE^3 : LK^3,

which was to be demonstrated.

167. Theorem.—*Any two similar polyedrons are to each other as the cubes of their homologous sides.* Dem.—No diagram is necessary for this demonstration. Let A be a solid angle of one polyedron, and *a* the homologous solid angle of the other. Since these solid angles are equal, we may suppose the less polyedron placed in the greater, so that the solid angles A and *a* shall coincide. Then if the greater polyedron be divided into pyramids, having their vertex in A, the planes which make these divisions, must evidently make corresponding divisions in the smaller. Thus the two similar polyedrons will be divided into the same number of similar pyramids, which by the preceding proposition, will be to each other as the cubes of their homologous sides (166). Hence a continued proportion might be formed, having the greater pyramids for its antecedents, the smaller pyramids for its consequents, and for its last ratio the cubes of two homologous sides of the two polyedrons. Then by adding the antecedents and consequents, excepting the last, we should have the greater polyedron to the less as the cubes of their homologous sides.

168. Theorem.—*Two similar cones or cylinders are to each other as the cubes of the radii of their bases.* Dem.—No diagram is necessary for this demonstration. Let C be one cone or cylinder, A its

altitude, and R the radius of its base; and let c, a, r, be corresponding expressions for the other cone or cylinder. Then (163)

$$C : c :: \pi \times R^2 \times A : \pi \times r^2 \times a :: R^2 \times A : r^2 \times a.$$

But by the definition (165)

$$A : a :: R : r.$$

Multiplying this, term by term, by the identical proportion $R^2 : r^2 :: R^2 : r^2$,

we have $R^2 \times A : r^2 \times a :: R^3 : r^3$.

Substituting this last ratio for its equal in the first proportion, we have $C : c :: R^3 : r^3$,

which was to be demonstrated.

APPENDIX.

CONTAINING AN ACCOUNT OF THE PRACTICAL APPLICATION OF SOME OF THE MOST IMPORTANT PRINCIPLES OF ELEMENTARY GEOMETRY, TOGETHER WITH QUESTIONS FOR THE EXERCISE OF THE LEARNER.

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169. We begin with the proposition of art. 17.—*Angles are measured by arcs of circles described from their vertices as centres.* Upon this proposition depend the construction and use of all the instruments, which have been invented for the measurement of angles in space, as well as for tracing them upon paper. The *protractor* and its use, we have already mentioned (18). The *quadrant* is an instrument used for measuring angles in a vertical plane. Its essential parts are represented in fig. 112. E D is a graduated arc of F 112
90° beginning at E. A C is a plumb-line attached to the vertex A. Near A and D are two *sight-holes* for determining accurately the direction of objects. The direction of a plumb-line A C, suspended freely, is called *vertical;* and the line F G, to which the vertical is perpendicular, is called *horizontal.* The angle H A G contained between the horizontal line and a line drawn to an object *above* it, is called the *angle of elevation* of the object: and the angle F A B, contained between the horizontal line and a line drawn to an object *below* it, is called the *angle of depression* of the object. Both these angles are readily measured by the quadrant. To find the angle of elevation H A G of an object H, the quadrant, kept always in a vertical plane by means of the plumb-line, is so placed that the object can be seen through the two sight-holes by the

eye placed at D. Then by counting the degrees from E to the plumb-line A C, we have the angle of elevation sought. For E A C=H A G (21,) each being complements of the same angle G A E. If the angle of depression F A B of an object B be required, the eye is placed at A and the line of the sight-holes directed to B. Then the degrees are counted from E to the plumb-line, as before. For E A C=F A B, each being complements of B A C.

170. Having spoken of *angles of elevation and depression* in connexion with the quadrant, the question naturally arises, for what purpose are these angles measured? The following example furnishes an answer. Standing at the distance of 100 feet from a tower situated upon a horizontal plane, it is proposed to find the height of the tower.—The angle of elevation measured by a quadrant, as directed in the preceding article, is found to be 40°. Then the height of the tower is found by the following geometrical construc-
F 113 tion, founded upon articles 55, 56. Let A (fig. 113) represent the place of observation, and A B the distance of the tower=100 feet. At A make an angle with the protractor=40°, the angle of elevation. This determines the direction of A C. Then as the tower is supposed to be perpendicular to the plane, erect a perpendicular at B to meet A C. B C will be the height of the tower, and its measure may be found by the same *scale of equal parts* by which A B was set off equal to 100. This example at the same time illustrates the importance of the proposition (55)—*a side and two adjacent angles determine the triangle.*

171. Having spoken in the preceding article of a *scale of equal parts*, it is proper that we explain its construction and use. To make a scale of equal parts, an inch or some other unit of length is taken as a basis, and as many of these units as may be desired, are accurately marked upon a rule. These may be again subdivided into halves and quarters. But the most important subdivision is the *decimal* and *centesimal* one, or that into tenths and hundredths. This we shall
F 114 explain by a diagram. Let A B (fig. 114) represent the inch or linear unit taken for the basis of the scale. Divide A B into 10 equal parts, by the method explained art. 73, and number the divisions as in the

figure. Thus we have *tenths* of an inch, or of any other unit represented by A B. Now, to find *hundredths*, construct upon A B the square A B C D, and divide each of the sides into 10 equal parts. Through the points of division of A D and B C, draw the horizontal lines parallel to A B, and number them on B C as in the figure. Then from the vertex A draw the oblique line A E to the first point of division in D C. Again join the first point of A B with the second of D C, the second of A B with the third of D C, and so on through the figure. By this construction we have a *scale of hundredths*, as may be easily shown. The triangles A D E and A G H are similar, having each a right angle and the angle at A common. Then

$$GH : DE :: AG : AD.$$

Now A G is 9 tenths of A D by construction ; hence G H is 9 tenths of D E. But D E is 1 tenth or 10 hundredths of an inch or of the unit taken for the basis. Therefore G H is 9 hundredths. By similar reasoning it might be proved that I K is 8 hundredths, P Q 7 hundredths, and so on to R S, which is one hundredth. If now it were required to find a number of hundredths greater than 10, as 34 for example, place one foot of the compasses at L in the horizontal line numbered 4, and extend the other to M in the oblique line numbered 3. L M will be 34 hundredths. For, as we have just seen, L T=4 hundredths, and T M=A 3 (38)= ,3 tenths=30 hundredths. Then L M=L T+T M= 34 hundredths. If it were required to find 76 hundredths, place one foot of the compasses at N in the horizontal line numbered 6, and extend the other to O in the oblique line numbered 7. N O will be 76 hundredths, which might be proved as before. In a similar manner we might find any number of hundredths from 1 to 99. This decimal and centesimal scale usually occupies the first place at the right hand of the scale of equal parts. And let it be observed that if A B represent 10 inches or units, each division of A B being an unit, the oblique divisions will be *tenths*. Also if A B represent a hundred inches, or units, each division of A B being 10, the oblique divisions will be *units*.

172. The following example will illustrate the use of the scale of equal and decimal parts.—The sides

and angles of a piece of land being found by measurement, it is proposed to draw a plan which shall represent their dimensions upon paper.—Let the number of the sides be 5, expressed in rods as follows. 1st=100, and makes an angle with the next=85°. 2d=110, angle with the next=109°. 3d=80, angle=136°. 4th=60, angle 100°. 5th=112, angle with 1st=110°. Now what we propose, is to form a reduced copy or plan of this field, which shall represent the sides and angles in their true proportions. This process is called *projecting* the field. We shall make the projection on the scale of 100 rods to an inch. We begin by making A B (fig. 115)=1 inch. At B with a protractor (18) we set off an angle=85°, which determines the direction of the next side B C. The length of B C is 110 rods, which, by the preceding article, is represented on the scale by 1 inch and 10 hundredths. The point C being thus determined, we make the angle at C=109° and for C D=80 rods, we take 80 hundredths from the scale. In this manner we proceed till the construction is completed; and the polygon A B C D E will represent the true proportions of the field. For by construction, it has its angles equal respectively to those of the field; and the sides have the same ratio to each other as the corresponding sides of the field. Therefore the polygon is similar to the field (85). It is obvious that perfect accuracy is not to be expected in measurements of this kind, where an error so small as one hundredth of an inch in a diagram, would amount to one rod in the actual dimensions of the field. Still less reliance could be placed upon constructions, in which, as is frequently the case, the scale of projection is 100 miles to an inch. But it is equally obvious that, with good instruments and great care in the use of them, such constructions may be regarded as very close approximations to the truth. If, for example, it were required to ascertain by the diagram the length of a straight line drawn from the beginning of the first side in the field to the end of the second, we find by taking A C in the compasses and applying it to the scale, that it is equal to 1 inch and 45 hundredths; and we thence conclude that the real line is 145 rods. But here an error of half a rod or 1 two hundredth of an inch could not be detected. Trigonometry furnishes methods

of obtaining more accurate results, but this belongs to another department of mathematics.

173. We shall now describe the instrument used for measuring angles upon a horizontal plane, such for example as those of the field in the preceding article. If it be required merely to measure the angles which the sides of a field, or which lines drawn to any two objects, make with each other, nothing more is necessary than a graduated circle, having a moveable index with sights. For by placing this at the vertex or angular point, so that the diameter from which the degrees are counted, shall coincide in direction with one of the sides, and then moving the index round till it coincides with the other side, the number of degrees at the index will express the measure of the angle. But it is usual to add to the graduated circle other appendages. The most important of these is a *magnetic needle*, the property of which is that for any given place, it preserves a constant position with respect to the meridian or North and South line, and if removed from this position, will immediately return to it. Another appendage is *a spirit level*, by which the horizontal position of the instrument is determined. Such an instrument is called a *Graphometer*, *Surveyor's Compass*, *Theodolite*, and *Mariner's Compass*, according to the varieties of its construction and appendages. It is used not only to determine the angles which lines make with each other, but the angles which lines make with the meridian, the position of which is known at each observation by the direction of the needle. This is the usual way of laying down the angles of a field. The course of a ship, too, is always determined by the angle which the direction of her keel makes with that of the needle in the compass. We shall only observe further concerning instruments, that lines on the surface of the earth are usually measured by a chain 4 rods long; and consisting of a hundred links, so as to be adapted to the decimal scale.

MENSURATION OF HEIGHTS AND DISTANCES.

174. After the foregoing explanations, it will be easy to understand the solution of the following prob-

lems in the *Mensuration of Heights and Distances;* all of which depend upon the properties of triangles demonstrated in articles 53, 54, 55, 56, 57, 58, 59, 60.

175. PROBLEM.—*To find the height of an accessible object standing on a horizontal plane.* At the distance of 200 feet from the bottom of a steeple, the angle of elevation of the top is found by the quadrant to be 47° 30′. Required the height of the steeple, that of the instrument being 5 feet. SOL.—From any
F 116 scale of equal parts set off A B (fig. 116)=200. Make an angle at A=47° 30′, the angle of elevation. This determines the direction of A C, as the right angle at B determines the direction of B C. The intersection of these two determines the length of B C. Take B C in the compasses and apply it to the same scale from which A B was taken. The length of B C is thus found to be 218, to which add 5, the height of the instrument, and we have 223 for the height of the steeple.

176. PROBLEM.—*To find the height of an accessible object standing on an inclined plane.* A tree standing on the declivity of a hill makes with the downward slope an angle=115°; and, at the distance of 250 feet down the hill, the angle made by a line drawn to the top of the tree with the upward slope =20°. Required the height of the tree. SOL.—Let
F 117 A B (fig. 117) represent the slope of the hill, and set it off from the scale=250. At B make an angle=115°. This determines the direction of B C. At A make an angle=20°. This determines the direction of A C. The intersection of A C and B C determines the height of the tree B C. Take B C in the compasses and apply it to the same scale from which A B was taken, and it will be found to be 121 feet.

177. PROBLEM.—*To find the height of an inaccessible object above a horizontal plane.* The angle of elevation of the top of a tree standing on the other side of a river=60°; and 100 feet farther distant, in the same vertical plane passing through the tree, the angle of elevation of the top is 40°. Required the height of the tree. SOL.—Draw the indefinite line
F 118 A D (fig. 118) to represent the horizontal plane. Take a point B for the first place of observation, and make the angle D B C=60°. This determines the

direction of B C. Then take B A from the scale=100, and at A make an angle=40°. This determines the direction of A C, and the intersection of A C and B C determines the point C, the top of the tree. From C let fall a perpendicular C D to the horizontal line, and this will be the height of the tree. Apply C D to the scale from which A B was taken, and it will give the height=162 feet.

178. PROBLEM.—*To find the distance between two objects on a horizontal plane, by observation made from the top of a tower or some other eminence whose height is known, the objects and the tower being in the same vertical plane.* From the top of a tower 143 feet above a level with the sea two ships are observed. The angle of depression of the first is 46°, and that of the other in a direct line beyond is 31°. Required the distance between the two ships. SOL.—Draw the line A B (fig. 119) to represent the height of the F 119
tower=143. At B erect the indefinite perpendicular B D to represent the horizontal line in which the ships are situated. Through A draw A E parallel to B D, for the purpose of setting off the angles of depression. Make the angles E A C=46°. The intersection of A C with B D determines C the position of the nearest ship. Then make the angle E A D=31°. The intersection of A D with B D determines D the position of the other ship. Apply D C to the scale and the distance of the two ships will be found=100 feet.

179. PROBLEM.—*To find the perpendicular distance of an inaccessible object, by observations made at two statioms.* At two stations 100 yards apart, the two bearings of a ship are found to be 53° and 79°. Required the perpendicular distance of the ship from the line joining the stations. *Note.*—By *bearing* we mean the angle contained between the line drawn to the ship and that drawn between the stations. SOL.—Take A (fig. 120) for one of the stations, and draw F 120
A B=100 yards. Make an angle at A=53°, and another at B=79°. These angles determine the directions of A C and B C, which by their intersection determine the point C the position of the ship. From C let fall the perpendicular C D. Apply C D to the scale, and the perpendicular distance will be found =105.

180. Problem.—*To find the distance between two inaccessible objects on a horizontal plane, by observations at two stations.* At two stations A and B
F 121 (fig. 121) 300 yards apart, two objects C and D were observed on the other side of a river. At A the bearings of the two objects were 58° 20′ and 95° 20′; and at B 53° 30′ and 98° 30′. Required the distance between the two objects. Sol.—Draw the line A B =300. Make the angle D A B=58° 20′ and the angle C A B=95° 20.′ These angles determine the directions of A D and A C. Again make the angle A B C=53° 30′ and the angle A B D=98° 30′. These angles determine the directions of B C and B D. The intersection of A C and B C determines the position of the object C, and the intersection of A D and B D determines the position of the object D. Take C D in the compasses and apply it to the scale from which A B was taken, and the distance of the two objects will be found=480 yards.

181. The following questions are proposed as an exercise in the foregoing problems.

1. At the distance of 200 feet from the bottom of a tower standing on a horizontal plane, the angle of elevation is 37°, the height of the instrument being 5 feet. What is the height of the tower?

2. Two persons at the distance of 200 feet from each other standing on the same level and in the same vertical plane passing through the top of a hill, find the angles of elevation at the two stations to be 48° and 27°. What is the height of the hill?

3. A leaning tower makes with the plane upon which it stands an angle of 85°; and at the distance of 175 feet from its base in the direction towards which it leans, the angle of elevation is 50°. What is the perpendicular height of the tower?

4. From an eminence 90 feet above a horizontal plane, the angles of depression of two objects in the same vertical plane with the observer, are found to be 30° and 50°. What is the distance between them?

5. At the extremities of a wharf 150 yards long, the bearings of a ship in the harbor are found to be 70° and 55°. What is the distance of the ship from the wharf?

6. From the extremities of a wharf 195 yards long

two ships are observed. At one extremity, their bearings are 40° and 92°; and at the other 45° and 101°. What is the distance between the ships?

182. In article 108 the following proposition was demonstrated—*The square of the hypothenuse of a right triangle, is equal to the sum of the squares of the other two sides.* This admits of important practical applications. We have already made use of it (113) in finding the ratio of the circumference of the circle to its radius or diameter. We then took it for granted that the student was acquainted with the process for *extracting the square root of numbers.* We shall do the same now, since the explanation of it is generally considered as belonging to arithmetic and algebra. Supposing then this knowledge, we have demonstrated that—*either side of a right triangle may be found, when the other two are known.* Thus in the right triangle A B C (fig. 122) $AC=(AB^2+BC^2)^{\frac{1}{2}}$. Substituting numbers, for A B 3 feet, and for B C 4 feet, we have $AC=(9+16)^{\frac{1}{2}}=5$. Hence the following rule—*To find the hypothenuse, add together the squares of the other two sides and extract the square root of their sum.* Again $AB=(AC^2-BC^2)^{\frac{1}{2}}$ and $BC=(AC^2-AB^2)^{\frac{1}{2}}$; or in numbers $AB=(25-16)^{\frac{1}{2}}=3$ and $BC=(25-9)^{\frac{1}{2}}=4$. Hence the following rule—*To find either side, subtract the square of the other side from the square of the hypothenuse, and extract the square root of the remainder.*

183. By applying the above rules, the following questions will be readily answered.

1. How long must a ladder be to reach the top of a wall 50 feet high, the foot of the ladder being 12 feet from the wall?

2. A prop 30 feet long has the upper end placed against the side of a building 16 feet from the ground. How far from the side of the house is the foot of the prop?

3. The height of a roof is 8 feet and the span 24. What is the length of the rafters? *Note.*—The span of the roof is the same as the breadth of the building.

4. A travels north 250 miles, and B from the same place travels east 300. How far are they apart?

184. In article 81 we demonstrated the following proposition—*If from any point without a circle a tangent and secant be drawn, the tangent is a mean proportional between the entire secant and the part without the circle.* This enables us to find the diameter of the earth by measurements made upon its surface; and, having found the diameter, it faciltates the measurement of certain heights and distances.

F 123 Thus let A F (fig. 123) represent the height of a mountain, F D the diameter of the earth, and A B the distance at which the summit of the mountain can be seen in the horizon. Then by the proposition referred to,

$$AF : AB :: AB : AD.$$

Multiplying the two means and dividing by the first extreme, $AD = AB^2 \div AF$. But $AD = AF + FD$. Then by subtracting A F, we have $FD = (AB^2 \div AF) - AF$. Hence the following rule for finding the diameter of the earth, when the height of a mountain, and the distance at which its summit can be seen in the horizon, are known. *Square the distance, divide by the height, and then subtract the height from the quotient.* Again from the first proportion we have $AB^2 = AD \times AF = (FD + AF) \times AF = FD \times AF + AF^2$. Then extracting the square root we have $AB = (FD \times AF + AF^2)^{\frac{1}{2}}$. Hence the following rule for finding the greatest distance at which an object of a known height is visible. *Multiply the diameter of the earth by the height of the object, to the product add the square of the height, and then extract the square root of the sum.* Lastly from the above equation $AF \times FD + AF^2 = AB^2$, which is an equation of the second degree, we have $AF = (\frac{1}{4} DF^2 + AB^2)^{\frac{1}{2}} - \frac{1}{2} DF$. Hence the following rule for finding the height of an object when we know the greatest distance at which it is visible. *To one fourth of the square of the diameter of the earth add the square of the distance, extract the square root of the sum, and from this root subtract half the diameter of the earth.*

185. The following questions may be solved by the rules demonstrated in the preceding article.

1. If a mountain be 3 miles high, and if its summit can be seen at sea, at the distance of 154 miles, what is the diameter of the earth ?

2. The diameter of the earth being known, how far can a mountain one mile high, be seen at sea ?

3. What is the height of an object which can be seen 30 miles at sea ?

4. If a ship's mast be 120 feet high, how far can its top be seen ?

5. If the top of a light-house be seen from the surface of the water at the distance of 15 miles, what is its height ?

MENSURATION OF SURFACES.

186. In articles 100, 101, 102, 103, 105, 106, 113, the following propositions were demonstrated.

1. *The area of a square is found by multiplying one of its sides by itself.*

2. *The area of any parallelogram is found by multiplying its base by its altitude.*

3. *The area of a triangle is found by multiplying its base by half its altitude.*

4. *The area of a trapezoid is found by multiplying its altitude by half the sum of its parallel sides.*

5. *The circumference of a circle is found by multiplying its diameter by* 3.1415926.

6. *The diameter of a circle is found by dividing its circumference by* 3.1415926.

7. *The area of a circle is found by multiplying its circumference by half its radius; or by multiplying the square of its radius by* 3.1415926.

8. *The area of a sector is found by multiplying its arc by half its radius.*

9. *The area of a segment is found by taking the difference between the area of a sector and triangle.* Note. —This rule supposes the segment to be less than a semicircumference. If it were greater, we should take the sum instead of the difference.

187. The following questions may be solved by applying the rules in the preceding article.

1. If the side of a square be ten feet, what is its area ?

2. If the area of a square be 225 square feet, what is the length of one of its sides ?

3. The side of a square piece of land is 80 rods. What number of acres does it contain ?

4. It is required to lay out a piece of land in the form of a square, which shall contain one acre. What must be the length of one of its sides ?

5. How many acres are there in a square mile ?

6. The base of a parallelogram is 40 feet and its altitude 16. What is its area ?

7. The area of a parallelogram is 144 square feet, and its base is 18 feet. What is its altitude ?

8. If a piece of land in the form of a parallelogram, have its base 180 rods and its altitude 70 rods, how many acres does it contain ?

9. If it be required to lay out 60 acres of land in the form of a parallelogram whose base is 120 rods, what must be its altitude ?

10. The base of a triangle is 15 inches, and its altitude 12 inches. What is its area ?

11. The area of a triangle is 4 square feet, and its altitude is 11 inches. What is its base ?

12. What is the difference between a triangle whose base is 10 feet and altitude 5 feet, and a parallelogram of the same base and altitude ?

13. What is the difference between a triangle whose base is 9 feet and altitude 8 feet, and a square whose side is 6 feet ?

14. One of the parallel sides of a trapezoid is 12 inches, and the other 16 inches, and the altitude 9 inches. What is the area ?

15. The area of a trapezoid is 70 square feet and its altitude 7 feet. What is the sum of its parallel sides ?

16. What is the difference between a trapezoid whose altitude is 20 feet and the sum of whose parallel side is 50 feet, and a triangle whose base is 100 feet and its altitude 10 feet ?

17. The diameter of a circle is 4 feet. What is its circumference ?

NOTE.—It is sufficient for all common purposes, to multiply by 3.1416 instead of 3.1415926.

18. The circumference of a circle is 75 feet. What is its diameter? What is its radius?

19. The radius of a circle is 7 feet. What is its area?

20. The circumference of a circle is 25 feet. What is its area?

21. The area of a circle is 100 square feet. What is its radius?

22. The area of a circle is 1000 square feet. What is its circumference?

23. What is the difference between a circle whose radius is 10 feet, and a triangle whose base is 10 feet and altitude 16 feet?

24. The radius of a sector is 4 feet and the arc 12 feet. What is the area?

25. The area of a sector is 90 square feet and the radius 8 feet. What is the length of the arc?

26. If the circumference of a circle is 27 feet, how long is an arc of that circle containing 60°.

NOTE.—This is found by the following proportion 360 : 60 : : 27 : answer (15).

27. If the radius of a sector is 5 feet and its arc 70°, what is its area?

28. If the radius of the sector A B C (fig. 124) is 3 feet, its arc B C 80°, and the chord B C 3.3 feet, what is the area of the segment?

NOTE.—The altitude A D of the triangle A B C is found by the equation $AD=(AB^2-\frac{1}{4}BC^2)^{\frac{1}{2}}$. For A D falls upon the middle of B C (28), and the square of $\frac{1}{2}$ B C is $\frac{1}{4}BC^2$.

188. In articles 149, 150, 151, 152, 153, the following propositions are demonstrated.

1. *The convex surface of a cylinder is found by multiplying the circumference of its base by its altitude.*
2. *The entire surface of a cylinder is found by adding the radius of the base to the altitude, and multiplying their sum by the circumference of the base.*
3. *The convex surface of a cone is found by multiplying the circumference of the base by half the side of the cone.*
4. *The entire surface of a cone is found by adding*

the radius of the base to the side of the cone, and multiplying their sum by half the circumfereuce of the base.

5. *The convex surface of the frustum of a cone is found by multiplying the side by half the sum of the greater and less circumferences.*

6. *The entire surface of the frustum of a cone is found by adding the side to the greater radius and multiplying the sum by half the greater circumference; then by adding the side to the less radius and multiplying the sum by half the less circumference; and lastly by adding these two products together.*

7. *The surface of a sphere is found by mutiplying the diameter by the circumference of a great circle.*

8. *The surface of a zone is found by multiplying its altitude by the circumference of a great circle.*

189. The following questions may be solved by applying the rules in the preceding article.

1. The radius of the base of a cylinder is 4 inches and its altitude 10 inches. What is its convex surface? What is its entire surface?

2. The area of the base of a cylinder is 20 square feet, and its altitude 8 feet. What is its entire surface? What is its convex surface?

3. The radius of the base of a cone is 7 inches and its side 16 inches. What is the convex surface? What is the entire surface?

4. The area of the base of a cone is 30 square feet and the altitude 10 feet. What is the entire surface? What is the convex surface?

NOTE.—The side of a cone is the hypothenuse of a right triangle, of which the altitude and the radius ofthe base are the other two sides. Thus A D the side of
F 108 the cone (fig. 108) is the hypothenuse of the right triangle A C D.

5. The greater radius of the frustum of a cone is 6 feet, the less radius 4 feet, and the side 7 feet. What is the convex surface? What is the entire surface?

6. The greater base of the frustum of a cone contains 40 square feet, the less base contains 25 square feet, and the altitude is 7 feet. What is the convex surface?

NOTE.—The side of the frustum of a cone is the hypothenuse of a right triangle, of which the altitude and the difference between the greater and less radii, are

the other two sides. Thus K D (fig. 108) is the hypothenuse of the right triangle K M D, of which K M =I C, is the altitude of the frustum, and M D=C D —I K, is the difference between the radii. F 108

7. The radius of a sphere being 8 feet, what is its surface ?

8. The diameter of the earth is nearly 7920 miles. Now supposing the figure of the earth to be perfectly spherical, how many square miles are there in its surface?

9. The circumference of a great circle of the earth is nearly 24880 miles, and the altitude of one of the frigid zones is nearly 320 miles, How many square miles are there in its surface?

10. The altitude of one of the temperate zones is nearly 2040 miles. How many square miles are there in its surface?

11. The altitude of the torrid zone is nearly 3200 miles. How many square miles are there in its surface?

MENSURATION OF SOLIDS.

190. In articles 139, 143, 145, 146, 155, 156, 157, 158, 159, 161, 162, the following propositions were demonstrated.

1. *The solidity of a cube is found by taking one of its sides three times as a factor.*

2. *The solidity of a prism or of a cylinder is found by multiplying the area of its base by its altitude.*

3. *The solidity of a pyramid or of a cone is found by multiplying the area of its base by one third of its altitude.*

4. *The solidity of the frustum of a pyramid or of a cone, is found by adding the solidities of three pyramids or cones of the same altitude as the frustum, and having for their respective bases, the greater base, the less base, and a mean proportional between the two.*

5. *The solidity of a sphere is found by multiplying its surface by one third of the radius.*

6. *The solidity of a spherical sector is found by multiplying the surface of the zone, which forms its base, by one third of the radius.*

7. *The solidity a spherical segment of one base is found by taking the difference or sum of the solidities of a sector and a cone; according as the segment is less or greater than a hemisphere.*

8. *The solidity of a spherical segment of two bases is found by taking the difference between the solidities of two spherical segments of one base.*

191. The following questions may be solved by the application of the rules in the preceding article.

1. If the side of a cube be 9 inches, how many cubic or solid inches does it contain?

2. How many cubic inches are there in a cubic foot?

3. If a cube contains 2550 solid feet, what is the length of its side?

NOTE.—This question supposes a knowedge of the process for extracting the cube root of numbers, the explanation of which is generally considered as belonging to arithmetic and algebra.

4. A cord of wood is in the form of a quadrangular prism, 8 feet long, 4 feet wide, and 4 feet high. How many solid feet does it contain?

5. If a prism contain 900 solid feet, and if its altitude be 20 feet, what is the area of its base?

6. What is the solidity of a pyramid whose base covers a thousand square feet, and whose altitude is 70 feet?

7. If a pyramid contain 800 solid feet and its base 50 square feet, what is its altitude?

8. If the greater base of the frustum of a pyramid be 75 square feet, its less base 60 square feet, and its altitude 20 feet, how many solid feet does it contain?

9. If the radius of the base of a cylinder be 10 feet and its altitude 20, how many solid feet does it contain?

10. If a cylinder contain 1000 solid feet and if the radius of its base be 6 feet, what is its altitude?

11. If the radius of the base of a cone be 8 feet and its altitude 30 feet, what is its solidity?

12. If the solidity of a cone be 2000 feet and the radius of its base be 10 feet, what is its altitude?

13. If the greater radius of the frustum of a cone be 9 feet, the less radius 6 feet and the altitude 12 feet, how many solid feet does the frustum contain?

14. If the radius of a sphere is 8 inches, what is its solidity?

15. How many cubic miles does the earth contain? How many cubic feet?

16. The diameter of the moon is 2160 miles. What is its volume or solidity?

17. The solidity or volume of the sun is 337102 times as great as that of the earth. What is the surface of the sun, supposing it spherical? What is its diameter?

NOTE. Since the solidity of a sphere (158) is equal to $\frac{4}{3}\pi R^3$, it follows that the radius is found by dividing the solidity by $\frac{4}{3}\pi$, and extracting the cube root of the quotient. Having found the radius, the surface is obtained by the rule before given, (188).

18. If the radius of a sphere be 6 feet, and the altitude of a zone forming the base of a spherical sector 2 feet, what is the solidity of the sector?

19. If the solidity of a spherical sector be 3000 solid feet, and its radius 50 feet, what is the surface of its zone or base?

20. If the radius H G (fig. 125) of a sphere be 8 feet, and the altitude F G of the segment P F G of one base, be 3 feet, what is the solidity of the segment P F G?

NOTE. The radius of the base of the segment P F is found thus. $PF = (HF^2 - HP^2)^{\frac{1}{2}}$. Now H F is the radius of the sphere, and H P is the radius of the sphere minus the altitude of the segment, or the altitude of the cone H F P.

21. If the radius H G (fig. 125) of a sphere be 8 feet, and the altitude O G of the segment O E G of one base, be 5 feet, how many solid feet does this segment contain? F 125

22. If the radius H G (fig. 125) of the sphere be 8 feet, and the altitude O P of the segment O E F P of two bases, be 8 feet, the greater base being at the distance of H O or three feet from the centre, what is the solidity of the segment O E F P?

23. If the radius of a sphere be 10 feet, and the altitude of a segment of two bases 4 feet; the greater base being 2 feet from the centre, and both in the same hemisphere; what is the solidity of the segment?

24. If the radius of a sphere be 12 feet, and the

altitude a segment of two bases 6 feet; the centre being between the bases, and one base being 4 feet from the centre; what is the solidity of the segment?

Note.—In finding the solidity of the greater segment of one base, the cone must here be added to the sector.

COMPARISON OF SIMILAR SURFACES AND SOLIDS.

192. In articles 116, 117, 164, 167, 168, the following propositions were demonstrated.

1. *Two similar polygons are to each other as the squares of their homologous sides.*
2. *Two circles are to each other as the squares of their radii or diameters.*
3. *The surfaces of two spheres are to each other as the squares of their radii.*
4. *The solidities of two spheres are to each other, as the cubes of their radii.*
5. *Two similar polyedrons are to each other as the cubes of their homologous sides.*
6. *Two similar cones or cylinders are to each other as the cubes of the radii of their bases.*

193. The following questions may be solved by the application of the rules in the preceding article.

1. The side of one triangle being 11 inches, and the corresponding side of a similar triangle being 3 inches, what is their ratio in numbers?
2. The dimensions of a field being found in rods, and the plan being projected upon the scale of 10 rods to an inch, what is the ratio of the plan to the field, expressed in numbers?
3. The engraving of a painting 10 feet square is made upon a surface 10 inches square. What is the ratio of reduction?
4. The homologous sides of two similar figures are as 8 to 5, and the area of the first is 120 square feet. What is the area of the second?
5. The radii of two circles are as 8 to 10. What is the ratio of the circles?
6. Two circles are to each other as 12 to 20. What is the ratio of their radii?
7. The radius of the earth is 3960 miles, and that

of Mars 2000 miles. What is the ratio of their surfaces ? What is the ratio of their solidities ?

8. The diameter of Jupiter is 89000 miles, what is the ratio of the surfaces of Jupiter and the earth? What is the ratio of their volumes ?

9. The sides of two similar polyedrons are to each as 3 to 9. What is the ratio of their solidities ?

10. A model of the temple of Minerva is made upon the scale of 6 feet to an inch. What ratio does the magnitude of the model bear to that of the original ?

11. The radii of two similar cylinders are to each other as 3 to 9. What is the ratio of their solidities?

12. The solidities of two similar cylinders are as 27 to 64. What is the ratio of their radii ?

13. The radii of two similar cones are to each other as 4 to 7. What is the ratio of their solidities ?

14. It is required to make a model or copy of a given cone, upon the scale of 16 feet to an inch. What ratio will the copy bear to the original ?

QUESTIONS FOR REVIEW.

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1. What is a point? Can you make or perceive a geometrical point?
2. What is a line? What are its extremities?
3. What is a straight line? Can you prove that it is the shortest distance between two points?
4. What is a linear unit?
5. Can the ratio or value of lines be expressed in numbers? How?
6. What is a broken line?
7. What is a curved line? What is it composed of?
8. What is the circumference of a circle? Radius? Diameter? Arc? Sector? Segment? Chord?
9. What is a degree? Minute? Second?
10. What is an angle? How is it read? How measured?
11. What is a right angle? Acute? Obtuse?
12. What is the sum of all the angular space about a point?
13. What is the supplement of an angle? Complement?
14. What are vertical angles? Are they equal? Why?
15. When is a line perpendicular to another? When oblique?
16. If a perpendicular be erected on the middle of a line what follows? What follows if that line be a chord?
17. Can you find the centre of a given arc? Can you find a circumference that will pass through any three points not in a straight line?

18. What measures the shortest distance from a point to a straight line ?
19. Can there be more than one perpendicular at a given point?
20. When are two lines parallel? Can they ever meet?
21. When two parallels meet a third line, how are the angles named? Why are they so called? and what is their property?
22. What are two interior angles on the same side, and what is proved of them?
23. What is proved of parallels comprehended between parallels?
24. What is proved of two angles which have their sides parallel and directed the same way?
25. What is proved of two parallel tangents or secants?
26. What is proved of every angle which has its vertex in the circumference?
27. What are inscribed angles, and what is proved of them?
28. What is a triangle? Can it always be inscribed? Why?
29. To what are the three angles always equal? Why?
30. Can a triangle have more than one right angle? Why?
31. What is a right triangle? What is the side opposite to the right angle?
32. Do two angles of a triangle determine the third angle?
33. What is an exterior angle, and what is proved of it?
34. What is an isosceles triangle, and what is proved of it?
35. What is an equilateral triangle, and what is proved of it?
36. What is proved of the greater side of every triangle?
37. What are the four cases in which two triangles are equal?
38. There are six things in a triangle; how many are necessary to determine the triangle?
39. Do three angles determine a triangle? Why?
40. What is a ratio? How is it written?

41. What is a proportion ? How is it written ? How is it read ?
42. What are the extremes ? Means ? Antecedents ? Consequents ?
43. Are the products of the means and extremes equal ?
44. What if two proportions have a common ratio ?
45. May the means or the extremes change places ?
46. May either ratio be multiplied or divided by the same number ?
47. May one proportion be multiplied by another or by itself ?
48. What is the ratio of the sum of the two first terms to the sum of the two last, and of the difference of the two first to that of the two last ?
49. In a continued proportion, what is the ratio of the sum of the antecedents to that of the consequents ?
50. What is proved of a line drawn through the sides of a triangle, parallel to the base ?
51. What are the problems that are solved upon this principle ?
52. What are similar triangles ? In what three cases are triangles similar ?
53. What important proposition is demonstrated of similar triangles ?
54. What is proved of a perpendicular let fall from the circumference to the diameter ?
55. What is proved of a tangent and secant drawn from the same point to a circle ?
56. What is meant by dividing a line in extreme and mean ratio ?
57. What is a figure of four sides called ?
58. What is a parallelogram ? Trapezoid ? Trapezium ?
59. What is a right parallelogram ? Square ? Oblong or Rectangle ?
60. What is an oblique parallelogram ? Rhombus ? Rhomboid ?
61. What is proved of the diagonal of a parallelogram ?
62. What is a polygon ? Regular ? Irregular ? Similar ?

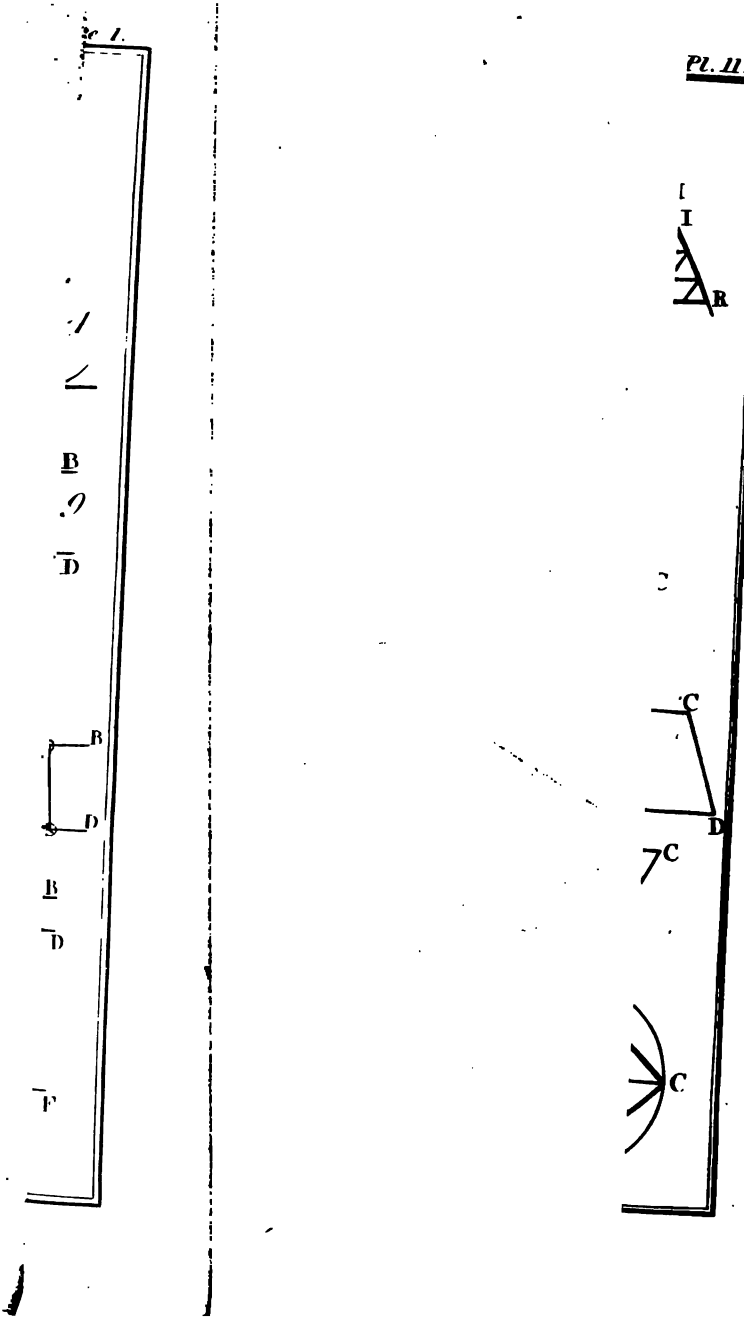
63. To what is the sum of the interior angles of a polygon equal? Why?
64. What is proved of two polygons composed of the same number of similar triangles?
65. What is proved of two regular polygons of the same number of sides?
66. Can every regular polygon be inscribed?
67. How is a square inscribed in a given circle?
68. How a regular hexagon? An equilateral triangle?
69. How a regular polygon of 10 sides? Of 15 sides? Of 5 sides?
70. What sort of a polygon is the circle demonstrated to be?
71. What is proved of the perimeters of regular polygons of the same number of sides?
72. What is the ratio of the circumferences of circles?
73. What is a surface? What are its boundaries?
74. How may we conceive it generated?
75. How many kinds of surfaces are there?
76. What is a plane surface? Polygonal? Curved?
77. What is the superficial unit? Why selected?
78. What is the meaning of area?
79. What is the area of a right parallelogram?
80. What is the area of a square?
81. What is the area of any parallelogram? Why?
82. What is the area of a triangle? Why?
83. What is the area of a trapezoid? Why?
84. What is the area of a regular polygon? Why?
85. What is the area of a circle? Why?
86. What is the area of sector? Segment? Why?
87. How do you find the area of an irregular polygon? Can it be converted into an equivalent triangle?
88. What is demonstrated of the square of the hypothenuse?
89. Can you make a square equivalent to the sum or difference of two given squares?
90. Can you make a parallelogram equivalent to a given square, and having the sum or difference of its base and altitude equal to a given line?
91 Can you make a square which shall be to a given square, in any given ratio?

92. Can you find the exact ratio of the circumference of a circle to its diameter ?
93. Can you find an approximate ratio ?
94. What is it ? Could the approximation be carried further ?
95. How do you find the circumference from the diameter ? How the diameter from the circumference ?
96. How do you find the area from the radius ?
97. What Greek character is used to express the above ratio ?
98. Can you make a square equivalent to any given figure ? How ? Why ?
99. How are surfaces compared ?
100 How are similar figures compared ? What is their ratio ?
101. What is the ratio of two circles ?
102. Do equal perimeters always enclose equal areas ? How is this proved ?
103. Among triangles of the same base and equal perimeters, which is the greatest ?
104. Among polygons of the same perimeter and number of sides, which is the greatest ?
105. Among polygons of equal perimeters and equal sides, which is the greatest ?
106. Among regular polygons of the same perimeter, which is greatest ? Why ?
107. Is a circle greater than any polygon of the same perimeter ? Why ?
108. How is the position of a plane determined ?
109. What is the intersection of two planes ?
110. How are plane angles measured ?
111. Do they have the same properties as linear angles ?
112. When is a line perpendicular to a plane ?
113. What measures the distance from a point to a plane ?
114. When are two planes, or a line and a plane, parallel ?
115. What is proved of parallel lines comprehended between parallel planes ?
116 What is proved of the intersections of two parallel planes by a third ?

117. What is proved of straight lines drawn between three parallel planes?
118. What is a solid? How generated? Boundaries?
119. What is a polyedron? What is the side or edge?
120. What are the planes which bound it called?
121. What is a prism? Its bases? Altitude? Convex surface? What is a right prism?
122. What is a parallelopiped? Right parallelopiped?
123. What is a cube?
124. What is a pyramid? Base? Altitude? Convex surface?
125. What is a regular pyramid? Frustum of a pyramid?
126. What are the three round bodies?
127. What is a cylinder? Base? Altitude? Convex surface?
128. What is a cone? Vertex? Altitude? Convex surface?
129. What is the side of a cone? What is the frustum of a cone?
130. What is a sphere? How generated?
131. What is every section made in the sphere?
132. What is a great circle? What a small one?
133. What is a spherical sector? Spherical segment?
134. What is the altitude of the sector or segment?
135. What is a zone? Its altitude?
136. How do you find the surface of a prism?
137. How do you find the surface of a pyramid
138. How do you find the surface of the frustum of a pyramid?
139. What is taken for the unit of solidity? Why?
140. What is the solidity of a right parallelopiped?
141. What is the solidity of a cube?
142. What is the solidity of any parallelopiped? Why?
143. What is the solidity of a right triangular prism? Why
144. What of any triangular prism? Of any prism? Why?
145. What is the solidity of a triangular pyramid? Why?
146. What is the solidity of any pyramid? Why?
147. What is the solidity of the frustum of a pyramid?

148. What is the solidity of a truncated triangular prism ?
149. What is the surface of a cylinder ? Convex surface ?
150. What is the surface of a cone ? Convex surface?
151. What is the surface of the frustum of a cone ? Convex surface ?
152. What is the surface of a sphere ? Of a zone ?
153. What ratio does the surface of an inscribed sphere bear to that of a circumscribed cylinder ?
154. What is the solidity of a cylinder ? Why ?
155. What is the solidity of a cone ? Why ?
156. What is the solidity of the frustum of a cone ? Why ?
157. What is the solidity of a sphere ? Why ?
158. What is the solidity of a spherical sector ?
159. What ratio does the solidity of an inscribed sphere bear to that of a circumscribed cylinder ;
160. How do you find the solidity of a spherical segment of one base ? Of two bases ?
161. How are solids compared ?
162. What is the ratio of the surfaces of two spheres ?
163. What is the ratio of the solidities of two spheres ?
164. What are similar polyedrons ?
165 What are similar cones and cylinders ?
166. What is the ratio of two similar pyramids ?
167. What is the ratio of two similar polyedrons ?
168. What is the ratio of two similar cones or cylinders ?

FINIS.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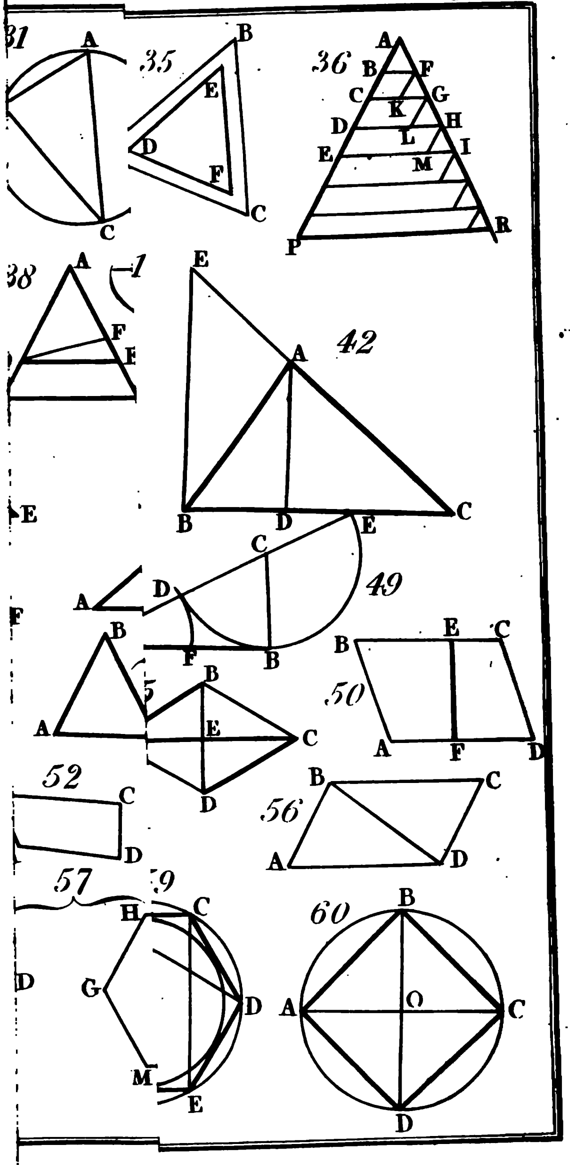

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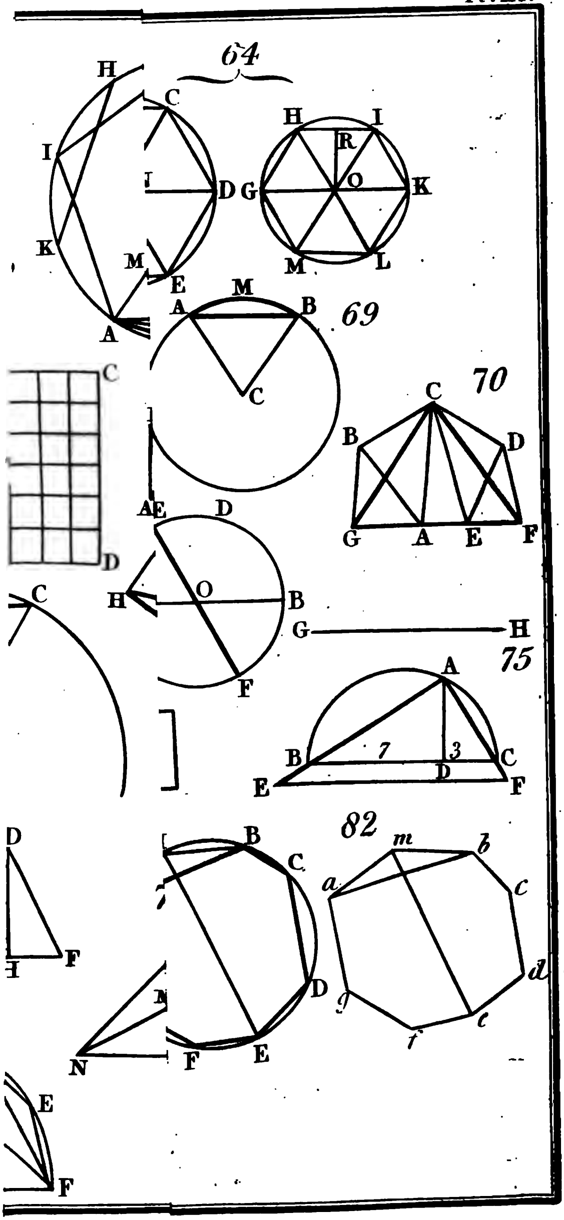

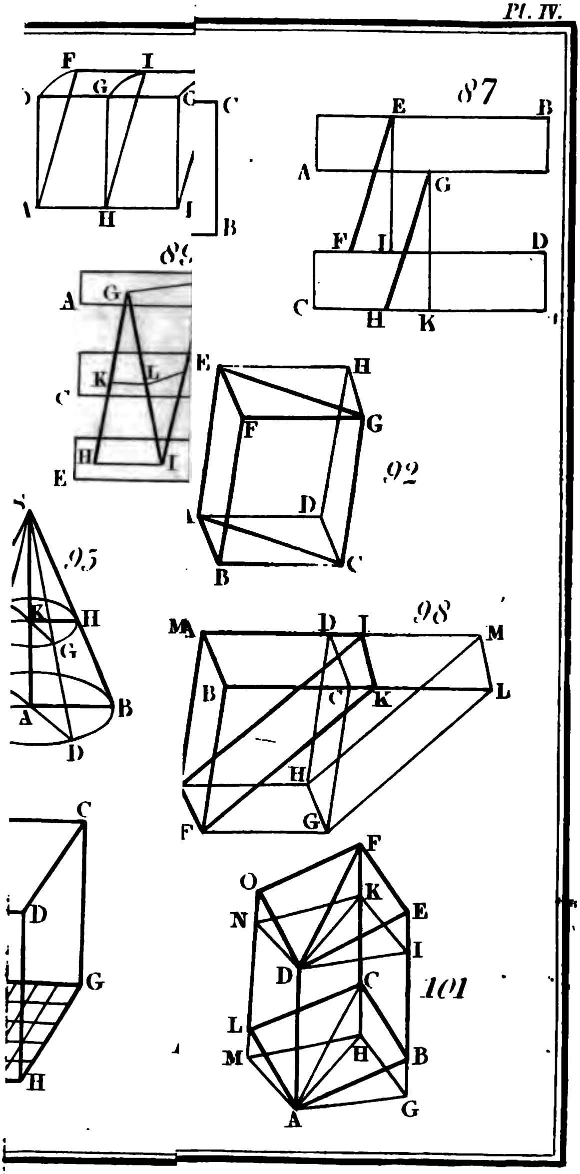
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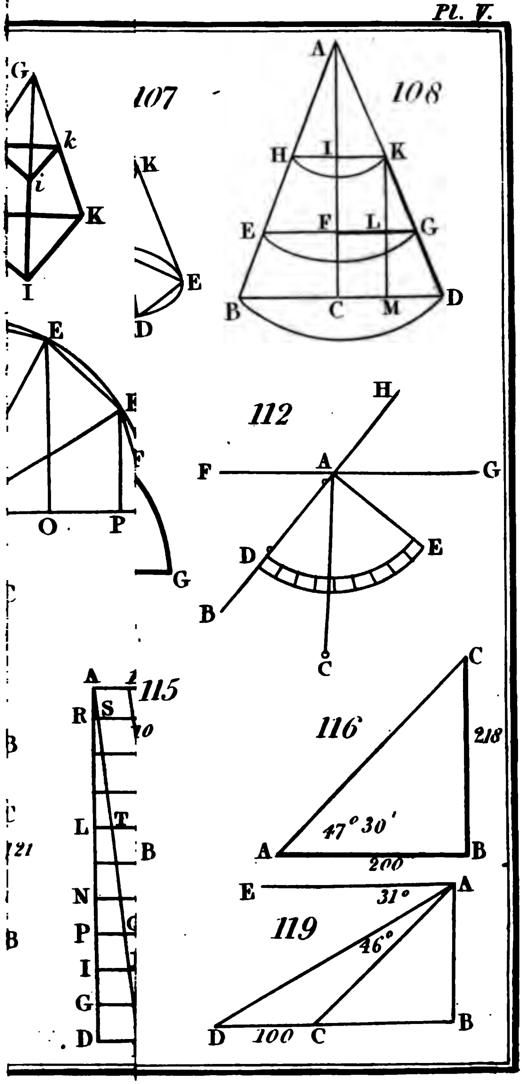

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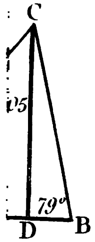
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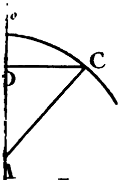
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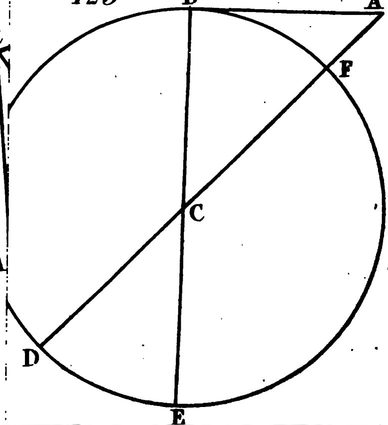
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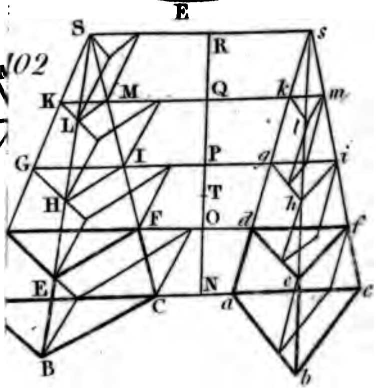
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